Coalgebraic Logic Programming

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Outline

1. Recursion and Corecursion
   - Inductive and Coinductive Types in Coq
   - Terminative and Productive Functions
   - Recursion and Corecursion without types

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2. Coalgebraic Logic Programming
3. Parallelism

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   - Terminative and Productive Functions
   - Recursion and Corecursion without types
2. Coalgebraic Logic Programming
3. Parallelism
4. Future directions: Applications to type inference

Outline

Today's talk...

...continuation of Thanos'es talk of yesterday:

- about logic programming (LP);
- about first order (= in Thanos'es terms infinite) language for LP;
- about how much we can merge methods of FP and LP...
- may be you will see some references to a possible game semantics.

Inductive Types and Recursive Functions

```
Inductive list (A : Type) : Type :=
 | nil : list A
 | cons : A -> list A -> list A.
```

Recursive functions have arguments of inductive types.

```
Fixpoint length (A:Type) (l: list A) : nat :=
match l with
 | nil => 0
 | _ :: l' => S (length l')
end.
```

Coinductive Types and Corecursive Functions

```
CoInductive stream (A:Set) :  Set :=
SCons:  A -> stream A -> stream A.
```

Corecursive functions have outputs of coinductive types. (Type of input arguments is not important.)

```
CoFixpoint map (s:Stream A) : Stream B :=
SCons (f (hd s)) (map (tl s)).
```

Termination

We require all computations to terminate, because of:

- Curry-Howard Isomorphism (propositions $\rightarrow\leftarrow$ types; proofs $\rightarrow\leftarrow$ programs): non-terminating proofs can lead to inconsistency.

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- Curry-Howard Isomorphism (propositions $\rightarrow\leftarrow$ types; proofs $\rightarrow\leftarrow$ programs): non-terminating proofs can lead to inconsistency.
- To decide type-checking of dependent types, we need to reduce expressions to normal form.

Productive Values

Values in co-inductive types are productive when all observations of fragments made using recursive functions are guaranteed to be computable in finite time.

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Values in co-inductive types are productive when all observations of fragments made using recursive functions are guaranteed to be computable in finite time.
The element of the stream at position n can be found by:

Definition

$$\begin{cases} \texttt{nth 0 (SCons a tl)} = \texttt{a} \\ \texttt{nth (S n) (SCons a tl)} = \texttt{nth n tl} \end{cases}$$

A given stream s is productive if we can be sure that the computation of the list `nth n s` is guaranteed to terminate, whatever the value of `n` is.

We call a function *productive at the input value i*, if it outputs a productive value at *i*.

Deciding Termination: Structural Recursion

A structurally recursive definition is such that every recursive call is performed on a structurally smaller argument.

In this way we can be sure that the recursion terminates.

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Example

```
Fixpoint length (A:Type) (l:  list A) : nat :=
match l with
 | nil => O
 | _ ::  l' => S (length l')
end.
```

Deciding Productivity: Guardedness

The guardedness condition insures that

* each corecursive call is made under at least one constructor;

** if the recursive call is under a constructor, it does not appear as an argument of any function.

Violation of any of these two conditions makes a function rejected by the guardedness test in Coq.

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CoFixpoint map (s:Stream A) : Stream B :=
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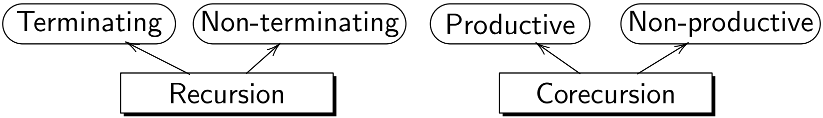
To notice:



- The role of types in definition of (co)recursive functions;
- The role of constructors and (co)-pattern matching;

Recursion and Corecursion in Logic Programming

Example

$$
\begin{aligned}
\texttt{nat(0)} &\leftarrow \\
\texttt{nat(s(x))} &\leftarrow \texttt{nat(x)} \\
\texttt{list(nil)} &\leftarrow \\
\texttt{list(cons x y)} &\leftarrow \texttt{nat(x), list(y)}
\end{aligned}
$$

Example

$$
\begin{aligned}
\texttt{bit}(0) &\leftarrow \\
\texttt{bit}(1) &\leftarrow \\
\texttt{stream(cons (x,y))} &\leftarrow \texttt{bit(x)},\texttt{stream(y)}
\end{aligned}
$$

SLD-resolution (+ unification and backtracking) behind LP derivations.

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$\mathtt{list(y)}$

$$\leftarrow \mathtt{list}(\mathtt{cons}(x, y))$$
$$|$$
$$\leftarrow \mathtt{nat}(x), \mathtt{list}(y)$$

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$$\leftarrow \texttt{nat}(\texttt{x}), \texttt{list}(\texttt{y})$$
$$|$$
$$\leftarrow \texttt{list}(\texttt{y})$$
$$|$$
$$\leftarrow \Box$$

The answer is x/O, y/nil, but we can get more substitutions by backtracking. We can backtrack infinitely many times, but each time computation will terminate.

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The answer is x/O, y/nil, but we can get more substitutions by backtracking. We can backtrack infinitely many times, but each time computation will terminate.

Nice, clean semantics: least Herbrand model exists, sound&complete, etc.: see Thanos'es Viva of yesterday.

Corecursion in LP?

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$\texttt{bit(0)} \leftarrow$

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No answer, as derivation never terminates.

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Example

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bit(0) ←
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stream(scons(x, y)) ←
            bit(x), stream(y)
```

No answer, as derivation never terminates.

Semantics may go wrong as well: least Herbrand models will contain an infinite term corresponding to stream: so completeness fails.

$$\leftarrow \mathtt{stream}(\mathtt{scons}(x, y))$$
$$|$$
$$\leftarrow \mathtt{bit}(x), \mathtt{stream}(y)$$
$$|$$
$$\leftarrow \mathtt{stream}(y)$$
$$|$$
$$\leftarrow \mathtt{bit}(x_1), \mathtt{stream}(y_1)$$
$$|$$
$$\leftarrow \mathtt{stream}(y_1)$$
$$|$$
$$\leftarrow \mathtt{bit}(x_2), \mathtt{stream}(y_2)$$
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$$\leftarrow \mathtt{stream}(y_2)$$
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$$\vdots$$

It can be worse....

Example

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list(nil) ←
```

No answer, as derivation never terminates.
Semantics goes wrong: this time, soundness!

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$$\leftarrow \texttt{bit}(x_2), \texttt{list}(y_2)$$
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$$|$$
$$\vdots$$

To notice:

- Distinction between (co)inductive type, (co)recursive function over (co)inductive type and a proof by (co)induction is erased.
- Without types guarding (co)recursion, things get messy:
  - ...not "just" termination, but also semantics
- We do not have a formalism to speak about termination and productivity, or generally, recursion/corecursion.

Note aside: LP has instances of dependent types, mixed induction/coinduction, recursion/corecursion....

Outline

CoALP: what is it about?

- syntactically – first-order logic programming;
- operationally – lazy (co)recursion;
- inspired by coalgebraic fibrational semantics;
- explores the tree-structure of partial proofs – "coinductive trees";
- uses lazy guarded corecursion using measures of corecursive steps given by coinductive trees (cf. "clocked corecursion");
- parallel...

Fibrational Coalgebraic Semantics of CoALP in 3 ideas

Operational View of Logic Programming

1. Let At be the set of all atoms appearing in a program P. Then P can be identified with a $P_f P_f$-coalgebra (At, p), where $p : At \longrightarrow P_f(P_f(At))$ sends an atom A to the set of bodies of those clauses in P with head A.

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2. Taking $p : At \longrightarrow P_f P_f(At)$, the corresponding $C(P_f P_f)$-coalgebra where $C(P_f P_f)$ is the cofree comonad on $P_f P_f$ is given as follows: $C(P_f P_f)(At)$ is given by a limit of the form

$$\ldots \longrightarrow At \times P_f P_f(At \times P_f P_f(At)) \longrightarrow At \times P_f P_f(At) \longrightarrow At.$$

This gives a "tree-like" structure: we call them $\&V$-trees.

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   This gives a "tree-like" structure: we call them $\&V$-trees.
3. For first order extension: Take *Lawvere Theory* $\mathcal{L}_\Sigma$ to model the signature Σ (objects are natural numbers, arrows – term arities, composition = substitution), and take $\mathcal{L}_\Sigma \rightarrow Set$ – to model (predicates in) At.

Examples

Program `Stream`: fibers are term arities. Take the fiber of 1. $\&V$-trees:

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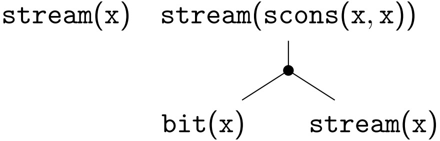

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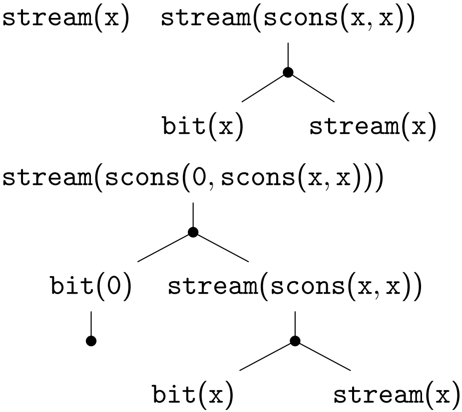

Computationally essential:

1. for coinductive `Stream` program, the $\&V$-trees are finite!!! – both in depth and in breadth;
2. each tree gives only a partial computation – it is not like eager SLD-trees we have seen earlier;
3. the effect of fibers is best modelled by restricting unification to term-matching (note resemblance to the pattern-matching in Functional setting).

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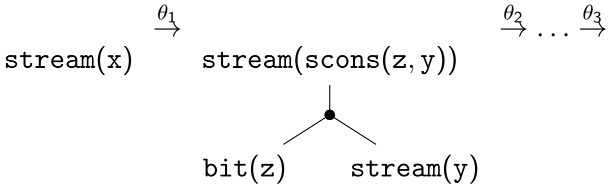
1. ⇒ gives hope for a formalism to describe termination and productivity
2. ⇒ hints there may be laziness involved...

Lazy Corecursion in CoALP: Coinductive trees

$$\texttt{stream(x)} \quad \xrightarrow{\theta_1}$$

Lazy Corecursion in CoALP: Coinductive trees



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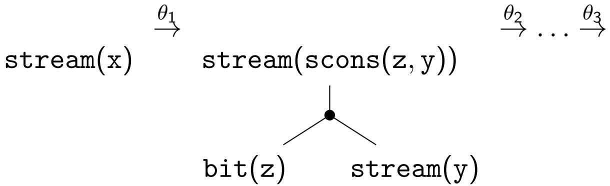
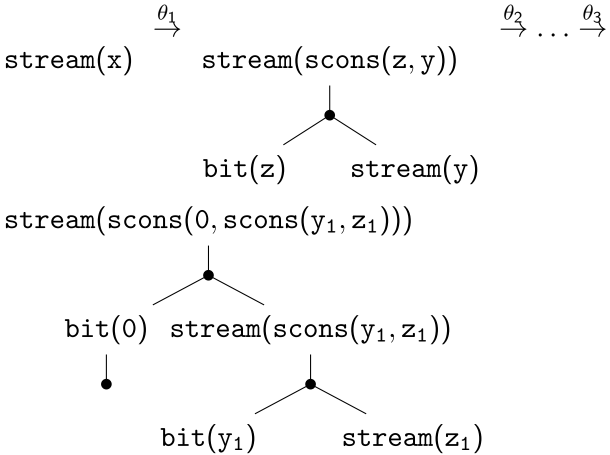


Note that transitions θ may be determined in a number of ways:

- using mgus;
- non-deterministically;
- in a distributed/parallel manner.

Lazy Corecursion in CoALP



The above would correspond to one-branch of SLD-derivations we have seen! The main driving force: separation of layers of computations into different dimensions.

Properties:

Komendantskaya, Power, Schmidt: Coalgebraic Logic Programming: from Semantics to Implementation, Journal of Logic and Computation, 2014.

- Sound and complete with respect to the coalgebraic semantcs;
- Finite computations are sound and complete with respect to the least Herbrand model semantics (so, we can do as much as standard Prolog for sure).
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What does it tell us beyond LP?

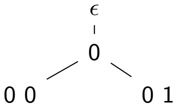
CoALP: the three-dimensional calculus of trees

1. Dimension 1: term-trees;
2. Dimension 2: coinductive trees;
3. Dimension 3: derivation trees.

Dimension-1: Term-trees

Take a tree-language $\mathbb{N}^*$.
Given an $L \in \mathbb{N}^*$, a term tree is a map $L \to \Sigma$, satisfying term arity restrictions.
Example:

0.



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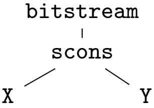


Notation:

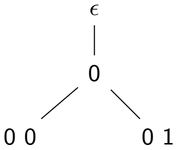
$\mathbf{Term}(\Sigma)$	*finite* term trees over Σ
$\mathbf{Term}^{\infty}(\Sigma)$	*infinite* term trees over Σ
$\mathbf{Term}^{\omega}(\Sigma)$	*finite and infinite* term trees over Σ

Dimension-2: Coinductive trees

Given an $L \in \mathbb{N}^*$, a coinductive tree is a map $L \to \mathbf{Term}(\Sigma_P)$, satisfying coinductive tree construction for P.

Example:

0.



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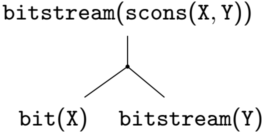

Notation:

$\mathbf{CTree}(\mathbf{Term}(\Sigma_P), P)$	all *finite* coinductive trees over $\mathbf{Term}(\Sigma_P)$
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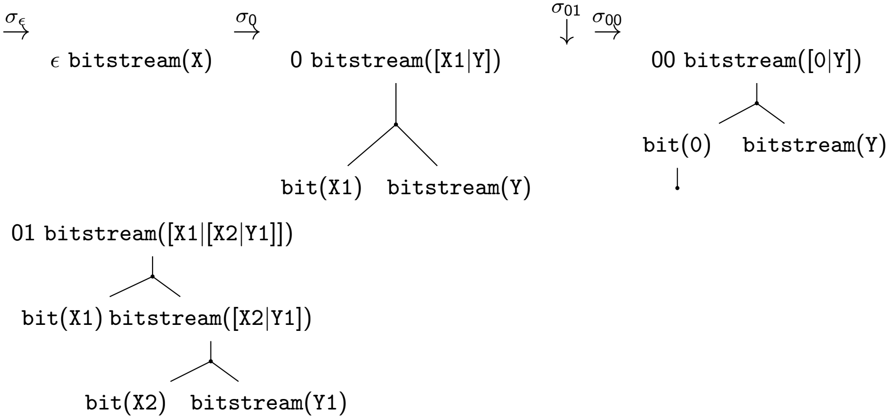
Dimension-3: Derivation trees

Given an $L \in \mathbb{N}^*$, a coinductive derivation is a map $L \to \mathbf{CTree}(\mathbf{Term}(\Sigma_P), P)$, satisfying the mgu requirement.



Dimension-3 notation

$\mathbf{CDer}(\mathbf{CTree}(\mathbf{Term}(\Sigma_P), P))$	all *finite* coinductive derivations over $(\mathbf{CTree}(\mathbf{Term}(\Sigma_P), P))$
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Theory of Productivity for LP

A first-order logic program P is *productive* if

for any term $t \in \mathbf{Term}(\Sigma_P)$, the coinductive tree CT with the root t belongs to $CTree(\mathbf{Term}(\Sigma_P), P)$.

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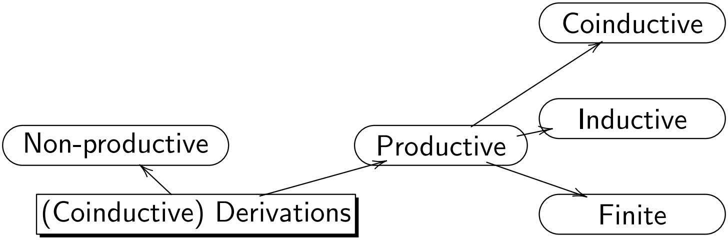
- inductive LPs all derivations for which are in $\mathbf{CDer}^{\omega}(\mathbf{CTree}(\mathbf{Term}(\Sigma_P), P))$; E.g. **ListNat**.
- coinductive LPs all derivations for which are in $\mathbf{CDer}^{\infty}(\mathbf{CTree}(\mathbf{Term}(\Sigma_P), P)))$ E.g. **Stream**.

Theory of Productivity in LP



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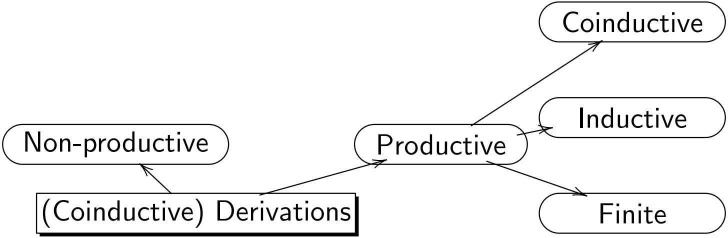
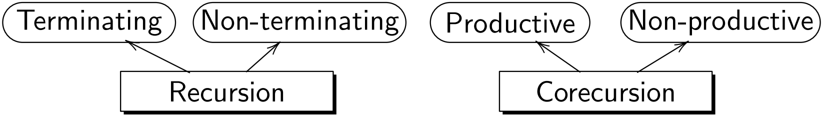
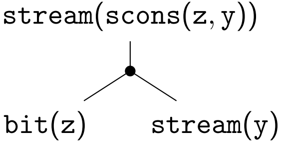


Compare with:



Deciding Productivity: Guardedness

- Dimension 1. Measures of reduction on term trees: `stream(y)` is a reduction of `stream(scons(x,y))`
- Dimension 2. Reduction on coinductive tree loops:



- Dimension 3. Discovery of derivation loops.

Example of guardedness issues

```
p(s(X1),X2,Y1,Y2) ← q(X2,X2,Y1,Y2)
q(X1,X2,s(Y1),Y2) ← p(X1,X2,Y2,Y2)
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$$p(s(x1), s(x2), s(y1), s(y2))$$
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$$\vdots$$

Soundness of Corecursion in LP

- CoALP is sound and complete for inductive programs;
- Soundness of coinductive programs is our next step.

Two directions:

- Imposing guardedness conditions, to ensure every coinductive tree is finite.
  To ban programs that are not guarded by constructors:
  `stream(scons x,y)` $\leftarrow$ `stream(scons x,y)`

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  To ban programs that are not guarded by constructors:

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  Unlike termination checks in Coq/Agda cannot be done fully statically (no types to help!), and needs some proof search in Dimension 3.

Soundness of Corecursion in LP

- CoALP is sound and complete for inductive programs;
- Soundness of coinductive programs is our next step.

Two directions:

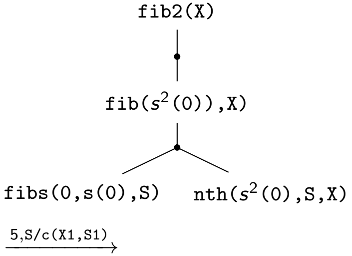
- Imposing guardedness conditions, to ensure every coinductive tree is finite.
  To ban programs that are not guarded by constructors:
  `stream(scons x,y)` $\leftarrow$ `stream(scons x,y)`
  Unlike termination checks in Coq/Agda cannot be done fully statically (no types to help!), and needs some proof search in Dimension 3.
- Determining when it is safe to make a coinductive conclusion (and finding a right coinductive hypothesis).
  (Again, the troubles come form un-typed setting.)

Stream of Fibonacci numbers:

Falls into infinite loops in Prolog and Prolog-like version of CoLP [Gupta et al. 2007] [Both are eager...] Those powerful SAT/SMT solvers would not do it either.

```
add(0,Y,Y).
add(s(X),Y,s(Z)) :- add(X,Y,Z).
fibs(X,Y,cons(X,S)) :- add(X,Y,Z), fibs(Y,Z,S).
nth(0,cons(X,S),X).
nth(s(N),cons(X,S),Y) :- nth(N,S,Y).
fib(N,X) :- fibs(0,s(0),S), nth(N,S,X).
fib2(X) :- fib(s(s(0)),X).
```

Examples of derivations with Fib: lazy step 1



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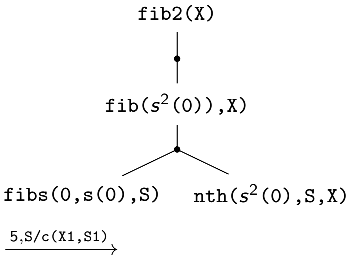


```
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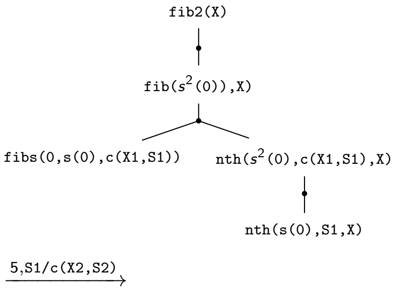
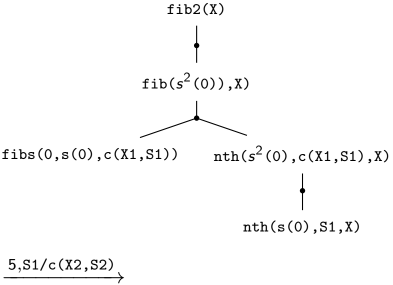
2.  add(s(X),Y,s(Z)) :-
add(X,Y,Z).
3.  fibs(X,Y,cons(X,S)) :-
add(X,Y,Z), fibs(Y,Z,S).
4.  nth(0,cons(X,S),X).
5.  nth(s(N),cons(X,S),Y) :-
nth(N,S,Y).
6.  fib(N,X) :- fibs(0,s(0),S),
nth(N,S,X).
7.  fib2(X) :- fib(s(s(0)),X).
```

Examples of derivations with Fib: lazy step 2



Examples of derivations with Fib: lazy step 2




```
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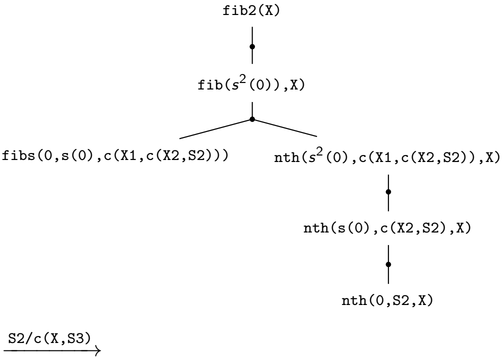
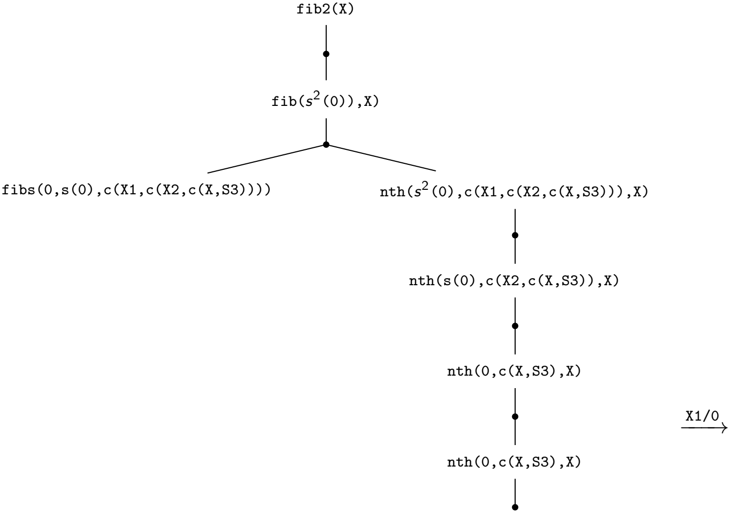
```

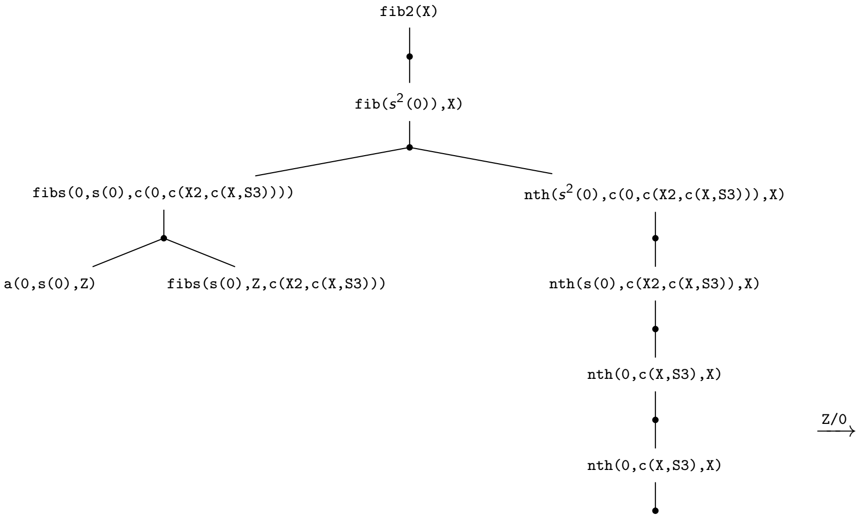
Examples of derivations with Fib: lazy step 3



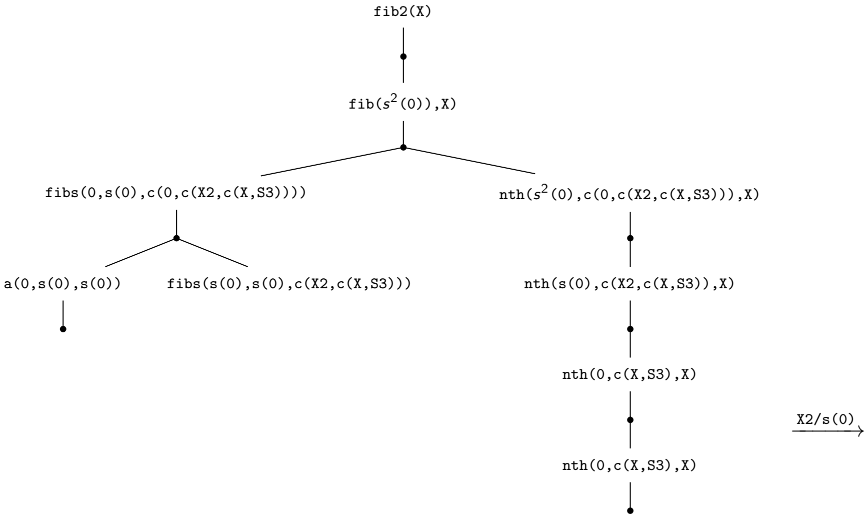
Examples of derivations with Fib: lazy step 4



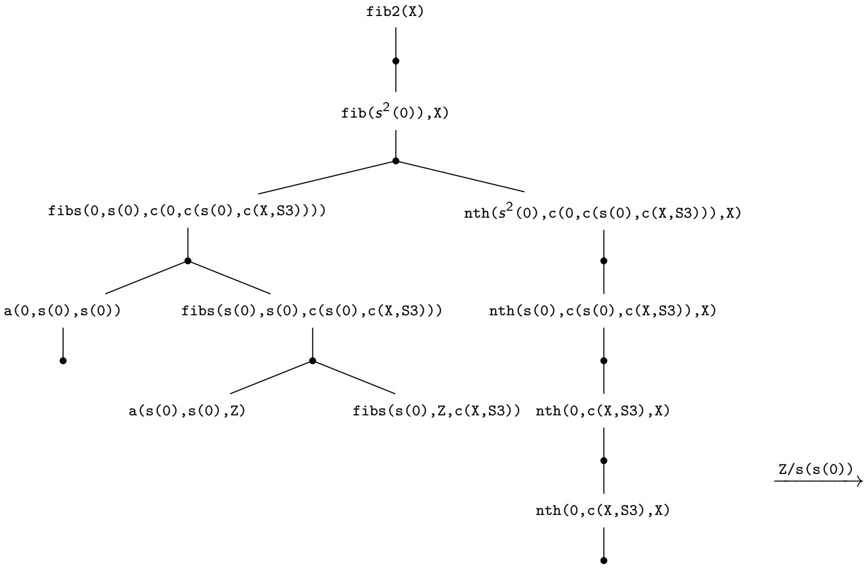
Examples of derivations with Fib: lazy step 5



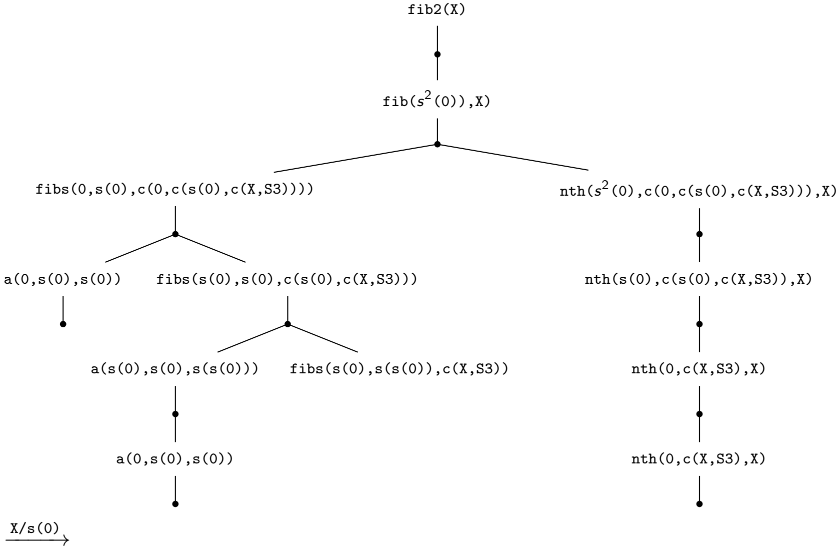
Examples of derivations with Fib: lazy step 6



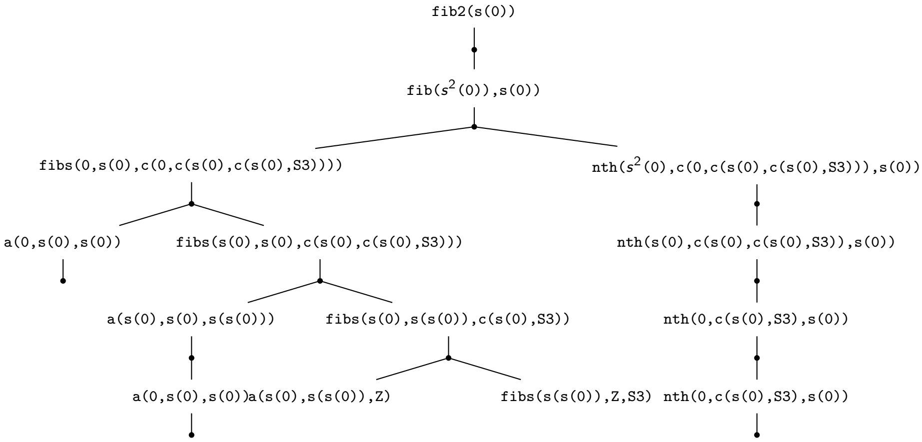
Examples of derivations with Fib: lazy step 7



Examples of derivations with Fib: lazy step 8



Examples of derivations with Fib: lazy step 9



Logic Programming dialects, compared

	Prolog	Parallel Prolog	Co-LP	CoALP
Fib example	No	No	No	Yes
Execution	Eager	Eager	Eager	Lazy
Corecursion				
Mode of execution				
Declarative semantics				
Operational semantics				

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Declarative semantics	lfp	lfp	gfp (restricted)	coalgebraic
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Operational semantics	transitions; states: lists of formulae	transitions; states: lists of formulae	transitions; states: lists of formulae	transitions; states: coinductive trees

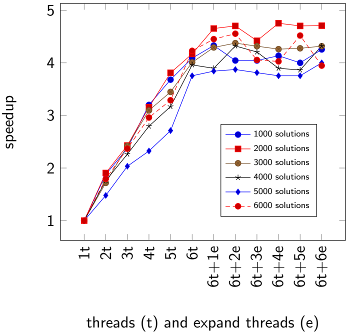
Outline

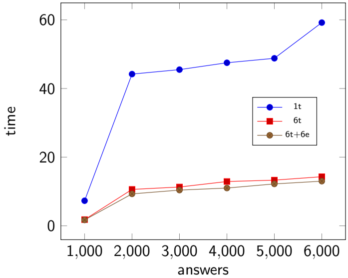
Parallelising CoALP

Komendantskaya, Schmidt, Heras: *Exploiting Parallelism in Coalgebraic Logic Programming*, ENTCS, 2014

```
1.  bit(0).
2.  bit(1).
3.  btree(empty).
4.  btree(tree(L,X,R)) ← btree(L), bit(X), btree(R).
```

Parallel CoALP





Directions we are exploring

Haskell implementation is nearly finished.

Current task: to find a "right" language to try CoALP-based type inference

- Using CoALP in Hume: for analysis of stream-based networks and/or for type inference;
- Type-inference in Haskell;
- SSReflect: overloading in canonical structures currently requires the use of back-tracking in LP-like algorithm. It could be parallel CoALP execution instead;
- CoALP for global type analysis in object-oriented languages: CoLP is already used for that.
- Formal Verification of CoALP-based type inference

The end

- Komendantskaya, Power, Schmidt: Coalgebraic Logic Programming: from Semantics to Implementation, Journal of Logic and Computation, 2014.
- Komendantskaya, Schmidt, Heras: *Exploiting Parallelism in Coalgebraic logic Programming*, ENTCS, 2014.
- A paper on implementing lazy guarded corecursion in CoALP using Haskell is in preparation...
- CoALP webpage has various prototype implementations to play with... http://staff.computing.dundee.ac.uk/katya/CoALP/

We will be happy to apply CoALP for TI (or other purposes) in *YOUR* language!

Milner, 1978

"A theory of Type Polymorphism in Programming"

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“A theory of Type Polymorphism in Programming”
An elegant match between polymorphic λ-calculus and type inference by means of Robinson’s unification/resolution algorithm.

Trend in type inference:

improvement in **expressiveness** of the underlying type system, e.g., in terms of

- *Dependent Types*,
- *Type Classes* [Wadler&Blott 89],
- *Generalised Algebraic Types* (GADTs) [Peyton Jones & al, 2006]
- *Dependent Type Classes* [Sozeau & al 08] and
- *Canonical Structures* [Gonthier& al 11].

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Milner-style decidable type inference does not always suffice (e.g. the *principal type* may no longer exist), and TI requires additional inference algorithms.

Implementations of new type inference algorithms include a variety of first-order decision procedures, notably Unification and Logic Programming (LP) [Peyton Jones & al, 2006], Constraint LP [Odersky Sulzmann, Vytiniotis & many more 1999-], LP embedded into interactive tactics (Coq's *eauto*) Sozeau & al. 08], and LP supplemented by rewriting [Gonthier & al, 11].

Motivation: type inference with Polymorphic types

List Length in Haskell

```
length ::  [a] -> Integer
length [] = 0
length (x:xs) = 1 + length xs
```

Logic program for type inference

$$
\begin{aligned}
&\texttt{cons}(\texttt{X}) && \leftarrow \texttt{X} = \texttt{Y} \rightarrow \texttt{list}(\texttt{Y}) \rightarrow \texttt{list}(\texttt{Y}).\\
&\texttt{plus}(\texttt{X}) && \leftarrow \texttt{X} = \texttt{int} \rightarrow \texttt{int} \rightarrow \texttt{int}.\\
&\texttt{nil}(\texttt{X}) && \leftarrow \texttt{X} = \texttt{list}(\texttt{Y}).\\
&\texttt{length}(\texttt{X}) && \leftarrow (\texttt{X} = \texttt{Y} \rightarrow \texttt{Z}) \ \& \ \texttt{nil}(\texttt{Y}) \ \& \ \texttt{Z} = \texttt{int} \ \& \ \texttt{cons}(\texttt{W}) \ \&\\
& && \quad (\texttt{W} = \texttt{W1} \rightarrow \texttt{W2} \rightarrow \texttt{Y}) \ \& \ \texttt{plus}(\texttt{U}) \ \&\\
& && \quad (\texttt{U} = \texttt{int} \rightarrow \texttt{Z} \rightarrow \texttt{Z}) \ \& \ \texttt{W2} = \texttt{Y}.
\end{aligned}
$$

Query: length(X)?

Answer (any existing PROLOG version): $X = list(\_) \rightarrow int$.

Trend to do more by type-inference:

... session types,

... writing contracts by means of types:

Example

Vytiniotis et al. "HALO: Haskell to Logic Through Denotational Semantics" [POPL'13]

```
f xs = head (reverse (True :  xs))
g xs = head (reverse xs)
```

Both f and g are well typed and "'can't go wrong"' in Milner's sense, but g will crash for empty list, and `f` will never crash.

Contract:

$$\texttt{reverse} \in (\texttt{xs} : \texttt{CF}) \rightarrow \{\texttt{ys} \mid \texttt{null xs} <=> \texttt{null ys}\}$$

Requires strong first-order type inference engines: Z3, Vampire, E...

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- Would it pay-off to get more conceptually elegant on type inference side? – especially bearing in mind the big emphasis on type inference in more expressive type systems.
- Would our "Coalgebraic Logic programming" grow to become a type-inference specific theorem prover (with stronger theoretical background and motivation than state-of-the-art SAT/SMT-solvers)?