Coalgebraic Logic Programming

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Recursion and Corecursion in Logic Programming

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3. 2000s: Gupta, Simon *et al*: CoLP: finite derivation procedure for coinductive programs, soundness and completeness for programs describing regular trees.
4. Our work, from 2010, – coalgebraic semantics for LP, and inspired derivation procedures.

Recursion and Corecursion in Logic Programming

Example

$$\begin{array}{rcl} \texttt{bit(0)} & \leftarrow & \\ \texttt{bit(1)} & \leftarrow & \\ \texttt{list(nil)} & \leftarrow & \\ \texttt{list(cons (X, Y))} & \leftarrow & \texttt{bit(X), list(Y)} \end{array}$$

Example

$$\texttt{stream(cons (X,Y))} \leftarrow \texttt{bit(X)}, \texttt{stream(Y)}$$

SLD-resolution (+ unification and backtracking) behind LP derivations.

Example

```
nat(0) ←
nat(s(x)) ← nat(x)
list(nil) ←
list(cons x y) ← nat(x),
                        list(y)
```

$$\leftarrow \mathtt{list}(\mathtt{cons}(x, y))$$
$$|$$
$$\leftarrow \mathtt{nat}(x), \mathtt{list}(y)$$

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$$\leftarrow \Box$$

The answer is x/O, y/nil, but we can get more substitutions by backtracking. We can backtrack infinitely many times, but each time computation will terminate.

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Nice, clean semantics: least Herbrand model exists, sound&complete, etc...

Corecursion in LP?

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No answer, as derivation never terminates.
Semantics may go wrong as well: least Herbrand models will contain an infinite term corresponding to stream: so completeness fails.

$$\leftarrow \texttt{stream}(\texttt{scons}(x, y))$$
$$|$$
$$\leftarrow \texttt{bit}(x), \texttt{stream}(y)$$
$$|$$
$$\leftarrow \texttt{stream}(y)$$
$$|$$
$$\leftarrow \texttt{bit}(x_1), \texttt{stream}(y_1)$$
$$|$$
$$\leftarrow \texttt{stream}(y_1)$$
$$|$$
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$$|$$
$$\vdots$$

It can be worse....

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list(nil) ←
```

No answer, as derivation never terminates.
Semantics goes wrong: this time, soundness!

$$
\begin{array}{c}
\leftarrow \texttt{list}(\texttt{cons}(x, y)) \\
| \\
\leftarrow \texttt{bit}(x), \texttt{list}(y) \\
| \\
\leftarrow \texttt{list}(y) \\
| \\
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| \\
\leftarrow \texttt{list}(y_1) \\
| \\
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| \\
\leftarrow \texttt{list}(y_2) \\
| \\
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\end{array}
$$

Solution - 1 [Gupta, Simon et al., 2007 - 2008]

If a formula repeatedly appears as a resolvent (modulo α-conversion), then conclude the proof.

Example

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|

$\leftarrow \texttt{bit(X1)}, \texttt{stream(X)}$

|

$\leftarrow \texttt{stream(X)}$

|

$\Box^{c}$

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|

$\square^c$

The answer is: $X/cons(0, X)$.

Requires programs to be regular, in order to be sound and complete

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- syntactically – first-order logic programming;
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- uses and-or parallel trees, but restricts unification to matching;

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Coinductive tree...

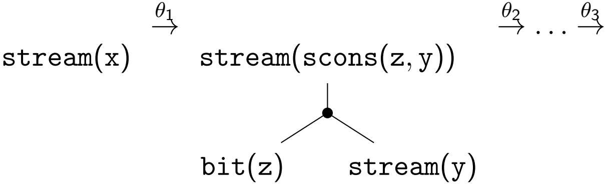
is an and-or-parallel tree in which unification is restricted to term-matching;

- Coinductive trees give a measure for lazy guarded corecursion, (cf. "clocked corecursion")

Lazy Corecursion in CoALP: Coinductive trees

$$\texttt{stream(x)} \xrightarrow{\theta_1}$$

Lazy Corecursion in CoALP: Coinductive trees



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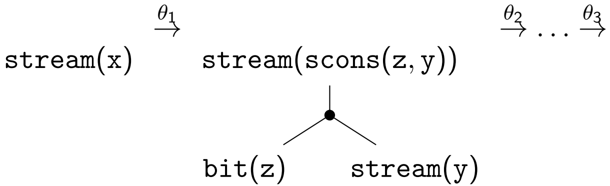
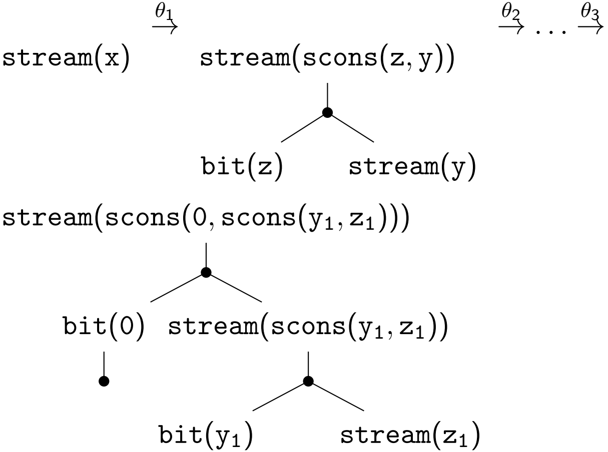


Note that transitions θ may be determined in a number of ways:

- using mgus;
- non-deterministically;
- in a distributed/parallel manner.

Lazy Corecursion in CoALP



The above would correspond to one-branch of SLD-derivations we have seen! The main driving force: separation of layers of computations into different dimensions.

Computationally essential:

1. for coinductive `Stream` program, the coinductive-trees are finite!!! – both in depth and in breadth;
2. each tree gives only a partial computation – it is not like eager SLD-trees we have seen earlier;

Computationally essential:

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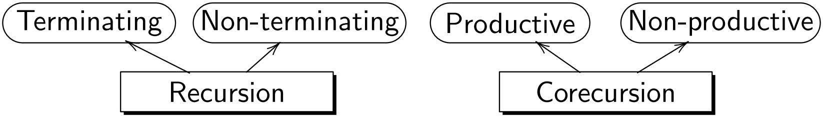
1. $\Rightarrow$ gives hope for a formalism to describe termination and productivity, as in functional languages
2. $\Rightarrow$ hints there may be laziness involved...

What do we gain?

1. A coherent theory of termination and productivity of recursion and corecursion in LP

Theory of Productivity in LP

Typeful functional theorem provers:



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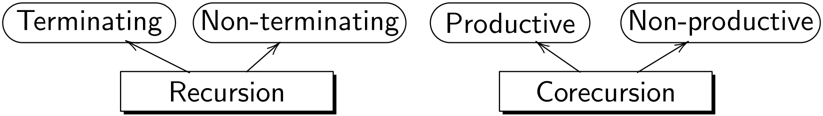
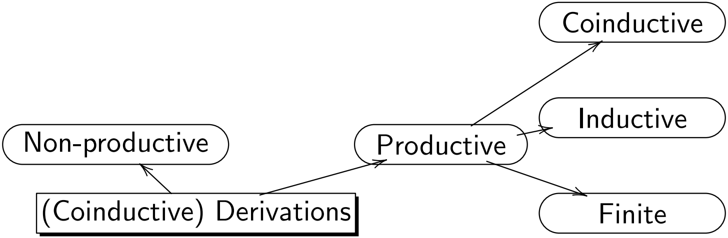


CoALP



What do we gain?

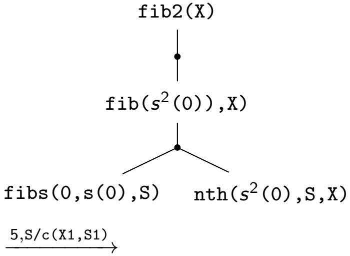
1. A coherent theory of termination and productivity of recursion and corecursion in LP
2. Extension of classes of inductive and coinductive programs we can handle,
3. Mixing induction/coinduction.

Stream of Fibonacci numbers:

Falls into infinite loops in Prolog and CoLP.

```
1.  add(0,Y,Y).
2.  add(s(X),Y,s(Z)) :- add(X,Y,Z).
3.  fibs(X,Y,cons(X,S)) :- add(X,Y,Z), fibs(Y,Z,S).
4.  nth(0,cons(X,S),X).
5.  nth(s(N),cons(X,S),Y) :- nth(N,S,Y).
6.  fib(N,X) :- fibs(0,s(0),S), nth(N,S,X).
7.  fib2(X) :- fib(s(s(0)),X).
```

Examples of derivations with Fib: lazy step 1



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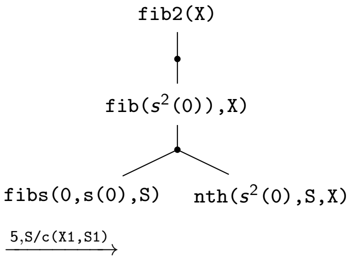


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```
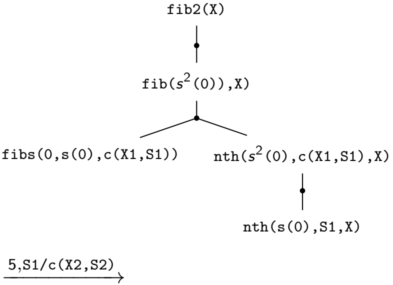
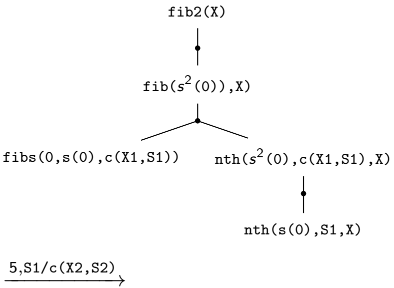

Examples of derivations with Fib: lazy step 2



Examples of derivations with Fib: lazy step 2

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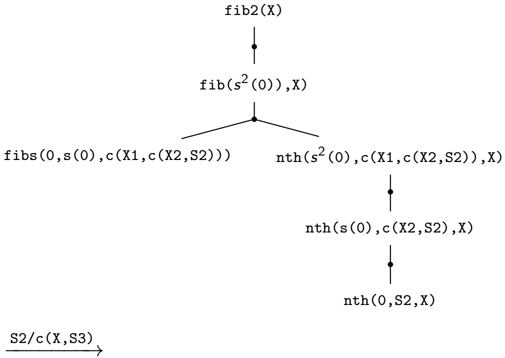
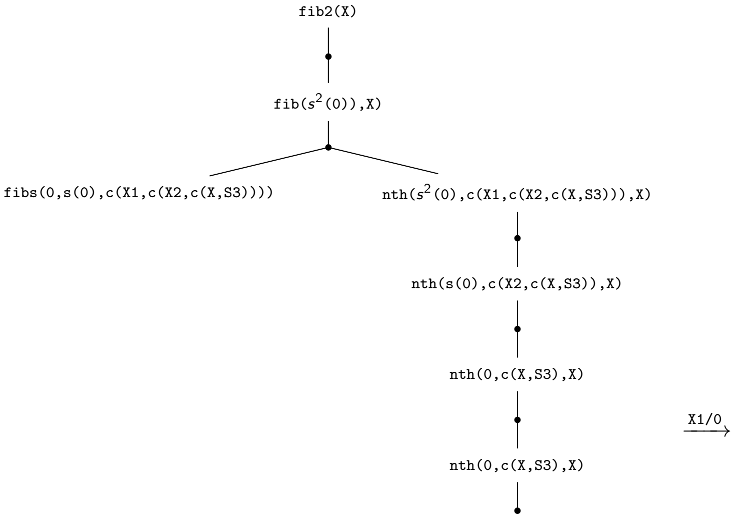
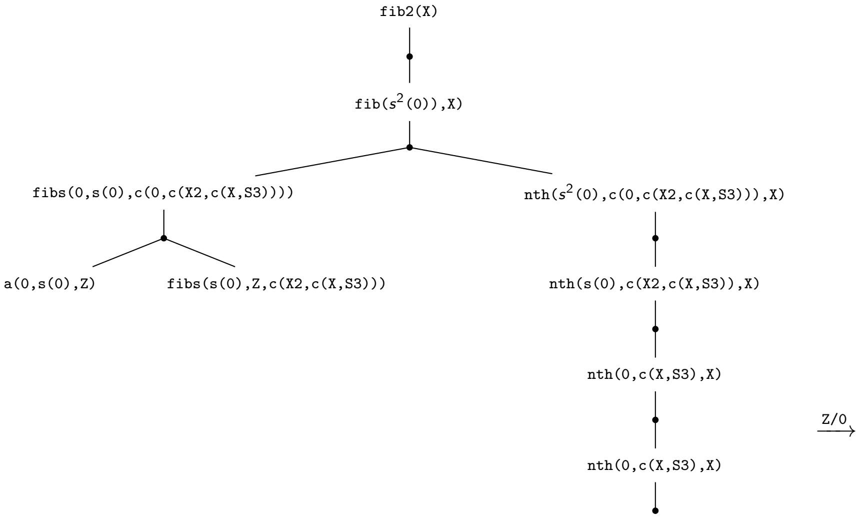
Examples of derivations with Fib: lazy step 3



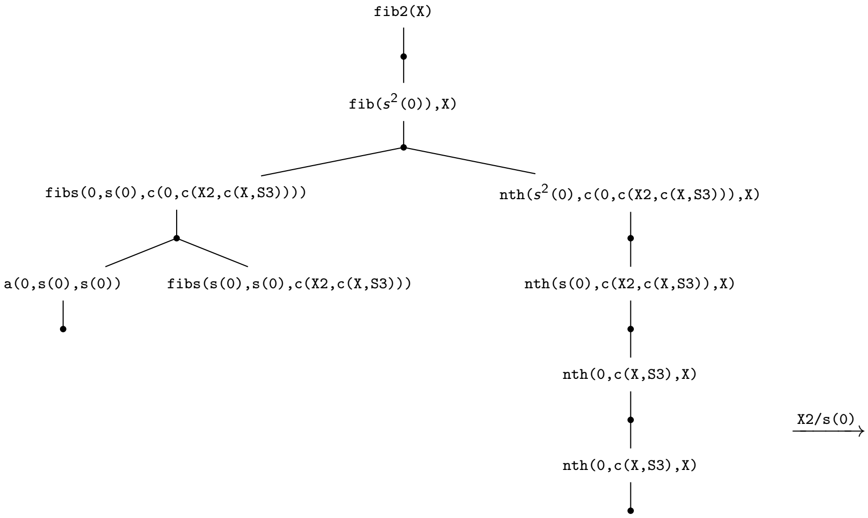
Examples of derivations with Fib: lazy step 4



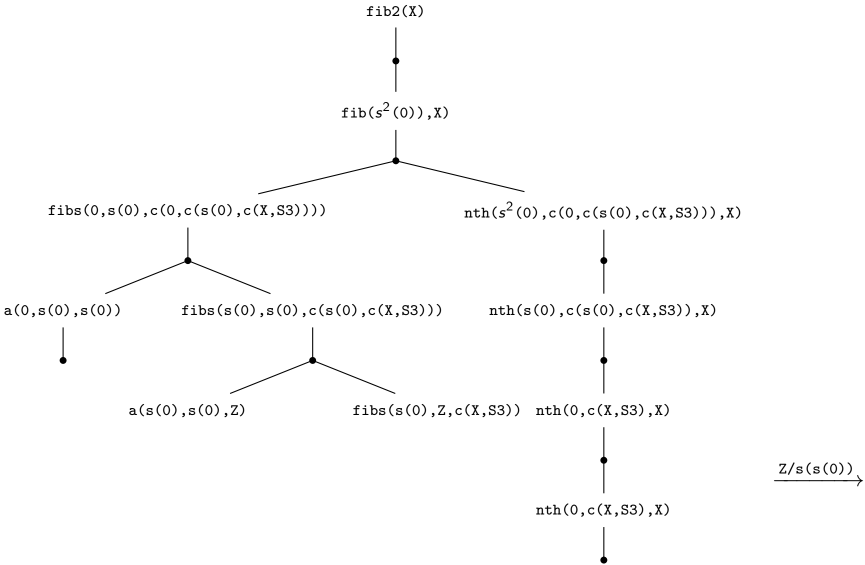
Examples of derivations with Fib: lazy step 5



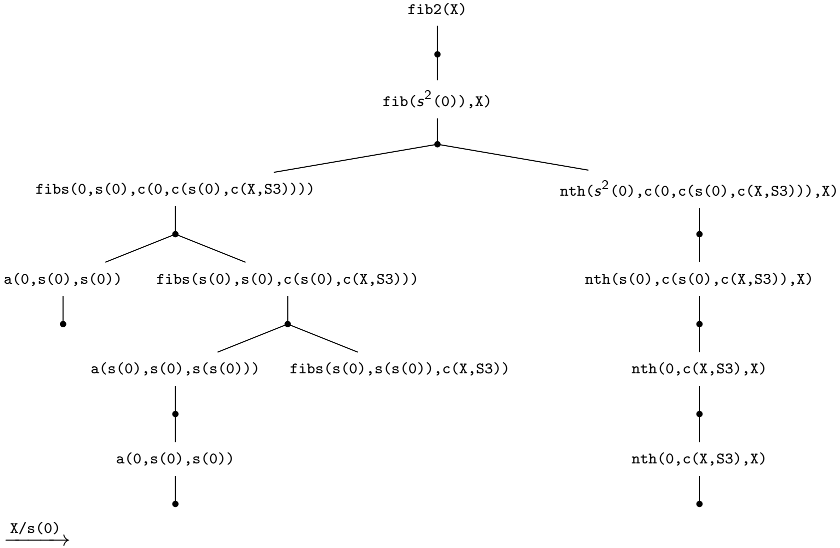
Examples of derivations with Fib: lazy step 6



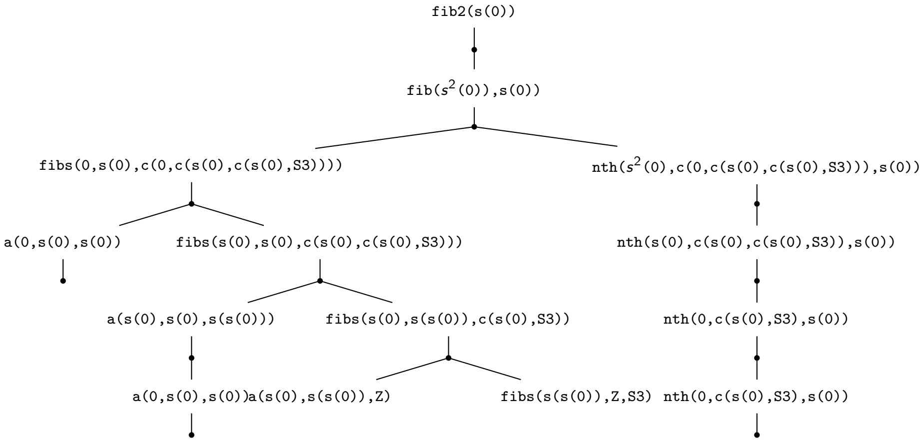
Examples of derivations with Fib: lazy step 7



Examples of derivations with Fib: lazy step 8



Examples of derivations with Fib: lazy step 9



CoALP Properties:

Komendantskaya, Power, Schmidt: Coalgebraic Logic Programming: from Semantics to Implementation, Journal of Logic and Computation, 2014.

- Sound and complete with respect to the coalgebraic semantcs;
- Finite computations are sound and complete with respect to the least Herbrand model semantics (so, we can do as much as standard Prolog).
- Adequacy result for observational semantics.

Logic Programming dialects, compared

	Prolog	Parallel Prolog	Co-LP	CoALP
Fib example	No	No	No	Yes
Execution	Eager	Eager	Eager	Lazy
Corecursion				
Mode of execution				
Declarative semantics				
Operational semantics				

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Mode of execution	Sequential	Parallel	Sequential	Parallel
Declarative semantics				
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Mode of execution	Sequential	Parallel	Sequential	Parallel
Declarative semantics	lfp	lfp	gfp (restricted)	coalgebraic
Operational semantics				

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Corecursion	No	No	by Regular Loop detection	Guardedness by constructors
Mode of execution	Sequential	Parallel	Sequential	Parallel
Declarative semantics	lfp	lfp	gfp (restricted)	coalgebraic
Operational semantics	transitions; states: lists of formulae	transitions; states: lists of formulae	transitions; states: lists of formulae	transitions; states: coinductive trees

Current and future work

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4. Extension of CoALP with constraints
5. Applications to type inference

... join us!

Thank you!

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