Using Analogical Representations for Mathematical Concept Formation

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Abstract. We argue that visual, analogical representations of mathematical concepts can be used by automated theory formation systems to develop further concepts and conjectures in mathematics. We consider the role of visual reasoning in human development of mathematics, and consider some aspects of the relationship between mathematics and the visual, including artists using mathematics as inspiration for their art (which may then feed back into mathematical development), the idea of using visual beauty to evaluate mathematics, mathematics which is visually pleasing, and ways of using the visual to develop mathematical concepts. We motivate an analogical representation of number types with examples of "visual" concepts and conjectures, and present an automated case study in which we enable an automated theory formation program to read this type of visual, analogical representation.

1 Introduction

1.1 The Problem of Finding Useful Representations

Antirepresentationalism aside, the problem of finding representations which are useful for a given task is well-known in the computational worlds of A.I. and cognitive modelling, as well as most domains in which humans work. The problem can be seen in a positive light: for instance, in "discovery",

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or "constructivist" teaching, a teacher is concerned with the mental representations which a student builds, tries to recognise the representations and provide experiences which will be useful for further development or revision of the representations [6]. As an example, Machtinger [22] (described in [6, pp. 94–95]) carried out a series of lessons in which groups of kindergarten children used the familiar situation of walking in pairs to represent basic number concepts: an even number was a group in which every child had a partner, and odd, a group in which one child had no partner. Machtinger encouraged the children to evolve these representations by physically moving around and combining different groups and partners, resulting in the children finding relatively interesting conjectures about even and odd numbers, such as:

even + even = even
even + odd = odd
odd + odd = even.

The problem of representation can also be seen in a negative light: for instance automated systems dealing in concepts, particularly within analogies, are sometimes subject to the criticism that they start with pre-ordained, human constructed, special purpose, frozen, apparently fine-tuned and otherwise arbitrary represenions of concepts. Thus, a task may be largely achieved via the *natural* intelligence of the designer rather than any *artificial* intelligence within her automated system. These criticisms are discussed in [2] and [20].

1.2 Fregean and Analogical Representations

Sloman [29, 30] relates philosophical ideas on representation to A.I. and distinguishes two types of representation: Fregean representations, such as (most) sentences, referring phrases, and most logical and mathematical formulae; and representations which are analogous – structurally similar – to the entities which they represent, such as maps, diagrams, photographs, and family trees. Sloman elaborates the distinction:

> Fregean and analogical representations are complex, i.e. they have parts and relations between parts, and therefore a syntax. They may both be used to represent, refer to, or denote, things which are complex, i.e. have parts and relations between parts. The difference is that in the case of analogical representations *both* must be complex (i.e. representation and thing) and there *must* be some correspondence between their structure, whereas in the case of Fregean representations there need be no correspondence. [30, p. 2]

In this paper we discuss ways of representing mathematical concepts which can be used in automated theory formation (ATF) to develop further concepts and conjectures in mathematics. Our thesis is that an analogical representation can be used to construct interesting concepts in ATF. We believe that analogical, pre-Fregean representations will be important in modeling Lakoff and Núñez's theory on embodiment in mathematics [19], in which a situated, embodied being interacts with a physical environment to build up mathematical ideas[1]. In particular, we believe that there is a strong link between analogical representation and visual reasoning.

The paper is organized as follows: we first consider the role of vision in human development of mathematics, and consider some aspects of the relationship between mathematics and the visual, in section 2. These include artists using mathematics as inspiration for their art (which may then feed back into mathematical development), the idea of using visual beauty to evaluate mathematics, mathematics which is visually pleasing, and ways of using the visual to develop mathematical concepts. In section 3 we motivate an analogical representation of number types with examples of "visual" concepts and conjectures, and in section 4 we present an automated case study in which we enable an automated theory formation program to read this type of visual representation. Sections 5 and 6 contain our further work and conclusions. Note that while we focus on the role that the visual can play in mathematics, there are many examples of blind mathematicians. Jackson [16] describes Nicolas Saunderson who went blind while still a baby, but went on be Lucasian Professor of Mathematics at Cambridge University; Bernard Morin who went blind at six but was a very successful topologist, and Lev Semenovich Pontryagin who went blind at fourteen but was influential particularly in topology and homotopy. (Leonhard Euler was a particularly eminent mathematician who was blind for the last seventeen years of his life, during which he produced half of his total work. However, he would still have had a normal visual system.)

2 The Role of Visual Thinking in Mathematics

2.1 Concept Formation as a Mathematical Activity

Disciplines which investigate mathematical activity, such as mathematics education and philosophy of mathematics, usually focus on proof and problem solving aspects. Conjecture formulation, or problem posing, is sometimes addressed, but concept formation is somewhat neglected. For instance, in the philosophy of mathematical *practice*, which focuses on what mathematicians do (as opposed to proof and the status of mathematical knowledge), Giaquinto gives an initial list of some neglected philosophical aspects of mathematical activity as discovery, explanation, justification and application, where the goals are respectively knowledge, understanding, relative certainty and practical benefits [12, p. 75]. Even in research on visual reasoning in mathematics education, visualizers are actually *defined* as people who "prefer to use visual methods when attempting mathematical *problems* which may be

[1] One of the goals of a project we are involved in, the *Wheelbarrow Project*, is to produce a computational model of Lakoff and Núñez's theory.

solved by both visual and non-visual methods" [24, p. 298] (our emphasis; see also [25, 26, 34] for similar focus) and mathematical giftedness and ability are measured by the ability to *solve* problems [17].

2.2 Problems with Visual (Analogical) Representations in Mathematics

Debate about the role of visual reasoning in mathematics has tended to follow the focus on proof, and centers around the controversy about whether visual reasoning – and diagrams in particular – can be used to prove, or to merely illustrate, a theorem. In contrast, in this paper we focus on the often overlooked mathematical skills of forming and evaluating concepts and conjectures. We hold that visual reasoning does occur in this context. For instance, the sieve of Eratosthenes, the Mandelbrot set and other fractals, and symmetry are all inherently visual concepts, and the number line and Argand diagram (the complex plane diagram) are visual constructs which greatly aided the acceptance of negative (initially called fictitious) and imaginary numbers, respectively.

The principle objection to using visual reasoning in mathematics, and diagrams in particular, is that it is claimed that they cannot be as rigorous as algebraic representations: they are heuristics rather than proof-theoretic devices (for example, [31]). Whatever the importance of this objection, it does not apply to us since we are concerned with the formation of concepts, open conjectures and axioms, and evaluation and acceptance criteria in mathematics. The formation of these aspects of a mathematical theory is not a question of rigor as these are not the sort of things that can be provable (in the case of conjectures it is the *proof* which is rigorous, rather than the formation of a conjecture statement). However, other objections may be relevant. Winterstein [35, chap. 3] summarizes problems in the use of diagrammatic reasoning: impossible drawings or optical illusions, roughness of drawing, drawing mistakes, ambiguous drawing, handling quantifiers, disjunctions and generalization (he goes on to show that similar problems are found in sentential reasoning). Another problem suggested by Kulpa [18] is that we cannot visually represent a theorem without representing its proof, which is clearly a problem when forming open conjectures. Additionally, in concept formation, it is difficult to represent diagrammatically the lack, or non-existence of something. For instance, primes are defined as numbers for which *there do not exist* any divisors except for 1 and the number itself. This is hard to represent visually. However, we hold that these problems sometimes occur because a diagrammatic language has been insufficiently developed, rather than any inherent problem with diagrams, and are not sufficient to prevent useful concept formation.

2.3 Automated Theory Formation and Visual Reasoning

We support our argument that visual reasoning plays a role in mathematical theory formation with a case study of automated visual reasoning in concept formation, in the domain of number theory. Although the automated reasoning community has focused on theorem proving, it is also well aware of the need to identify processes which lead to new concepts and conjectures being formed, as opposed to solely proving ready made conjectures. Although such programs [8, 11, 21, 27] include reasoning in the domains of graph theory and plane geometry – prime candidates for visual reasoning – representation in theory formation programs has tended to be algebraic.

2.4 Relationships between Mathematics and Art

Artists who represent mathematical concepts visually develop a further relationship between mathematics and the visual. For instance, Escher [10] represented mathematical concepts such as regular division of a plane, superposition of a hyperbolic plane on a fixed two-dimensional plane, polyhedra such as spheres, columns, cubes, and the small stellated dodecahedron, and concepts from topology, in a mathematically interesting way. Other examples include Albrecht Dürer, a Renaissance printmaker and artist, who contributed to polyhedral literature [7]; sculptor John Robinson, who displayed highly complex mathematical knot theory in polished bronze [1]; and the artist John Ernest (a member of the British constructivist art movement), who produced art which illustrates mathematical ideas [9], such as his artistic representation of the equation for the sum of the first n natural numbers $1+2+3+...+n = n(n+1)/2$, a Möbius strip sculpture and various works based on group theory (Ernest's ideas fed back into mathematics as contributions to graph theory).

2.5 Using the Visual to Evaluate Mathematical Concepts

A visual representation of mathematical concepts and conjectures can also suggest new ways of evaluation. As opposed to the natural sciences, which can be evaluated based on how they describe a physical reality, there is no obvious way of evaluating mathematics beyond the criteria of rigor, consistency, etc. Aesthetic judgements provide a further criterion. [28] examines the roles of the aesthetic in mathematical inquiry, and describes views by Hadamard [13], von Neumann [32] and Penrose [23], who all argue that the motivations for doing mathematics are ultimately aesthetic ones. There are many quotes from mathematicians concerning the importance of beauty in mathematics, such as Hardy's oft-quoted "The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics." [14, p. 85]. Clearly, beauty may not be visual (one idea of mathematical beauty is a "deep theorem", which establishes connections between previously unrelated domains, such as Euler's identity $e^{i\pi} + 1 = 0$). However, aesthetic judgements of beauty can often be visual: Penrose [23], for example, suggests many visual examples of aesthetics being used to guide theory formation in mathematics.

3 Visual Concepts and Theorems in Number Theory

3.1 Figured Number Concepts

Figured numbers, known to the Pythagoreans, are regular formulations of pebbles or dots, in linear, polygonal, plane and solid patterns. The polygonal numbers, for instance, constructed by drawing similar patterns with a larger number of sides, consist of dot configurations which can be arranged evenly in the shape of a polygon [15, 33]. In Figure 1 we show the first three triangular numbers, square numbers and pentagonal numbers. These provide a nice example of Sloman's analogical representations.

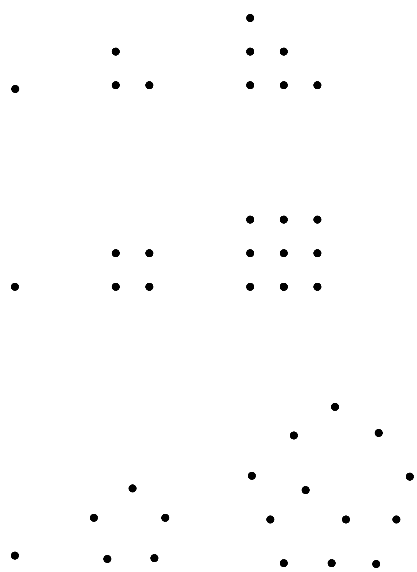

Fig. 1 Instances of the first three polygonal numbers: triangular numbers 1–3, square numbers 1–3 and pentagonal numbers 1–3.

The first six triangular numbers are $1, 3, 6, 10, 15, 21$; square numbers $1, 4, 9, 16, 25, 36$; and pentagonal numbers $1, 5, 12, 22, 35, 51$. Further polygonal numbers include hexagonal, heptagonal, octagonal numbers, and so on. These concepts can be combined, as some numbers can be arranged into multiple polygons. For example, the number 36 can be arranged both as a square and a triangle. Combined concepts thus include square triangular numbers, pentagonal square numbers, pentagonal square triangular numbers, and so on.

3.2 Conjectures and Theorems about Figured Concepts

If interesting statements can be made about a concept then this suggests that it is valuable. One way of evaluating a mathematical concept, thus, is by considering conjectures and theorems about it. We motivate the concepts of triangular and square numbers with a few examples of theorems about them. Firstly, the formula for the sum of consecutive numbers, $1 + 2 + 3 + ... + n = n(n + 1)/2$, can be expressed visually with triangular numbers, as shown in Figure 2.

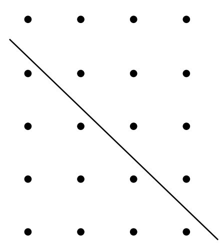

Fig. 2 Diagram showing that $1+2+3+4 = 4*5/2$. This generalizes to the theorem that the sum of the first n consecutive numbers is $n(n + 1)/2$.

Secondly, Figure 3 is a representation of the theorem that the sum of the odd numbers gives us the series of squared numbers:

$$\begin{aligned}
&1 = 1^2\\
&1 + 3 = 2^2\\
&1 + 3 + 5 = 3^2\\
&1 + 3 + 5 + 7 = 4^2\\
&\ldots
\end{aligned}$$

Finally, Figure 4 expresses the theorem that any pair of adjacent triangular numbers add to a square number:

$$1 + 3 = 4$$
$$3 + 6 = 9$$
$$6 + 10 = 16$$
$$45 + 55 = 100$$
$$\ldots$$

Further theorems on polygonal numbers include the square of the nth triangular number equals the sum of the first n cubes, and all perfect numbers are triangular numbers.

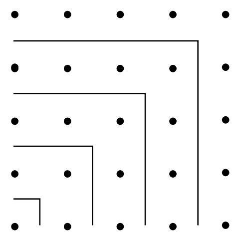

Fig. 3 Diagram showing that $1+3+5+7+9 = 5^2$; This generalizes to the theorem that the sums of the odd numbers gives us the series of squared numbers.

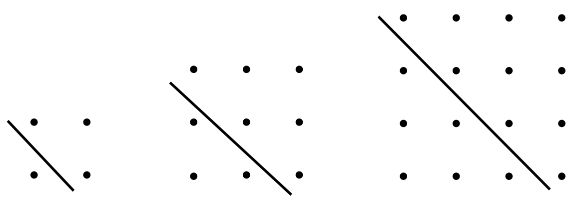

Fig. 4 Diagram showing that $S_2 = T_1 + T_2$; $S_3 = T_2 + T_3$; $S_3 = T_2 + T_3$; This generalizes to the theorem that each square number (except 1) is the sum of two successive triangular numbers.

4 An Automated Case Study

4.1 The HR Machine Learning System

The HR machine learning system[2] [3] takes in mathematical objects of interest, such as examples of groups, and core concepts such as the concept of being an element or the operator of a group. Concepts are supplied with a definition and examples. Its concept formation functionality embodies the constructivist philosophy that "new ideas come from old ideas", and works

[2] HR is named after mathematicians Godfrey Harold Hardy (1877–1947) and Srinivasa Aiyangar Ramanujan (1887–1920).

by applying one of seventeen production rules to a known concept to generate another concept. These production rules include:

- The *exists* rule: adds existential quantification to the new concept's definition
- The *negate* rule: negates predicates in the new definition
- The *match* rule: unifies variables in the new definition
- The *compose* rule: takes two old concepts and combines predicates from their definitions in the new concept's definition

For each concept, HR calculates the set of examples which have the property described by the concept definition. Using these examples, the definition and information about how the concept was constructed and how it compares to other concepts, HR estimates how interesting the concept is, [4], and this drives an agenda of concepts to develop. As it constructs concepts, it looks for empirical relationships between them, and formulates conjectures whenever such a relationship is found. In particular, HR forms equivalence conjectures whenever it finds two concepts with exactly the same examples, implication conjectures whenever it finds a concept with a proper subset of the examples of another, and non-existence conjectures whenever a new concept has an empty set of examples.

4.2 Enabling HR to Read Analogical Representations

When HR has previously worked in number theory [3], a Fregean representation has been used. For example the concept of integers, with examples $1-5$ would be represented as the predicates: integer(1), integer(2), integer(3), integer(4), integer(5), where the term "integer" is not defined elsewhere. In order to test our hypothesis that it is possible to perform automated concept formation in number theory using analogical representations as input, we built an interface between HR and the Painting Fool [5], calling the resulting system HR-V (where "V" stands for "visual"). The Painting Fool performs colour segmentation, thus enabling it to interpret paintings, and simulates the painting process, aiming to be autonomously creative. HR-V takes in diagrams as its object of interest, and outputs concepts which categorise these diagrams in interesting ways.

As an example, we used the ancient Greek analogical representation of figured numbers as dot patterns. We gave HR-V rectangular configurations of dots on a grid for the numbers 1–20, as shown in Figure 5 for 7, 8 and 9. Rearranging n dots as different rectangular patterns and categorizing the results can suggest concepts such as:

- *equality* (two separate collections of dots that can be arranged into exactly the same rectangular configurations),
- *evens* (numbers that can be arranged into two equal rows),
- *odds* (numbers that cannot be arranged into two equal rows),

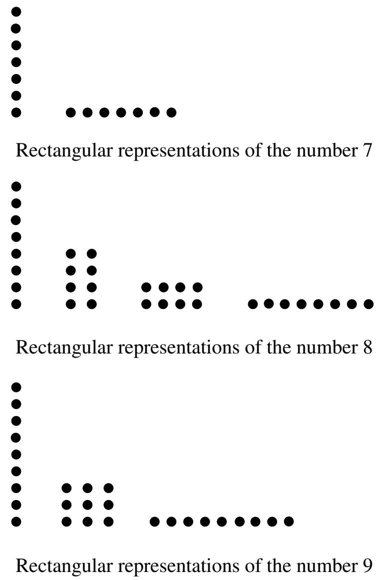

Fig. 5 Rectangular configurations for the numbers 7, 8 and 9.

- *divisors* (the number of rows and columns of a rectangle),
- *primes* (a collection of dots that can only be arranged as a rectangle with either 1 column or 1 row),
- *composite numbers* (a collection of dots that can be arranged as a rectangle with more than 1 column and more than 1 row), and
- *squares* (a collection of dots that can be arranged as a rectangle with an equal number of rows and columns).

HR-V generated a table describing the rectangle concept, consisting of triples: [total number of dots in the pattern, number of rows in the rectangle, number of columns in the rectangle]. This is identical to the divisor concept (usually input by the user) and from here HR-V used its production rules to generate mathematical concepts. For instance, it used $match[X, Y, Y]$, which looked through the triples and kept only those whose number of columns was equal to the number of rows, and then $exists[X]$ on this subset, which then omitted the number of rows and number of columns and left only the first part of the triple: number of dots. This resulted in the concept of square numbers (defined as a number y such that there exists a number x where $x * x = y$) In another example, HR-V counted how many different rectangular configurations there were for each number – $size[X]$ – and then searched through this list to find those numbers which had exactly two rectangular configurations – $match[X, 2]$ – to find the concept of prime number. In this way, HR-V categorized the type or shape of configuration which can be made from different numbers into various mathematical concepts.

4.3 Results

After a run of 10,000 steps, HR-V discovered evens, odds, squares, primes, composites and refactorable numbers (integers which are divisible by the number of their divisors), as shown in Figure 6, with a few dull number types defined.

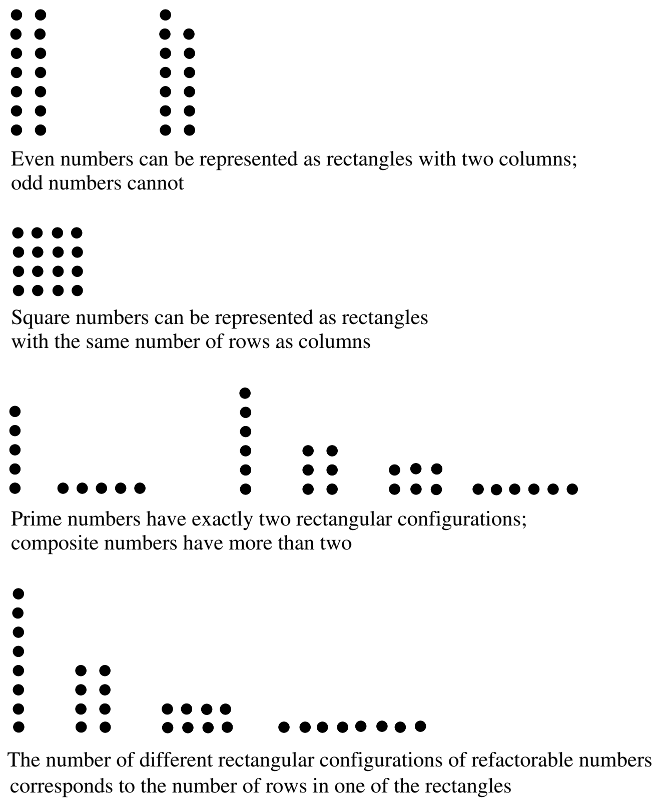

Fig. 6 The concepts which HR-V generated from rectangular configurations of numbers 1–20: evens, odds, squares, primes, composites and refactorables.

Thus, HR-V is able to invent standard number types using number rectangle definitions of them. This is a starting point, suggesting that automatic concept formation is possible using analogical representations. We would like to be able to show that analogical representations can lead to concepts which would either not be discovered at all, or would be difficult to discover, if represented in a Fregean manner.

5 Further Work

Although the representations can be seen as analogous to the idea they represent, they are still subject to many of the criticisms described in Section 1, such as being pre-ordained, human constructed, special purpose and frozen. In order to address this, we are currently enabling HR to produce its own dot patterns, via two new production rules. These production rules are different to the others in that they produce a new model or entity from an old one (or old ones), as opposed to the other seventeen production rules which produce a new concept from an old concept (or old concepts). In both production rules, an entity is a configuration of dots which is represented by a 100*100 grid with values of on/off where "on" depicts a dot.

The first new production rule will take in one entity and add dots to produce a new one. There are two parameters:

(i) the type of adding (where to put the dot);
(ii) the numerical function (how many dots to add).

The second new production rule will take in two entities and merge them to produce a new one. This is like the "compose" production rule but takes entities rather than concepts. Alternatively, it can take in one entity and perform symmetry operations on it.

While we have focused on concept formation in this paper, we believe that analogical representations, and visual reasoning in particular, can be fruitful in other aspects of mathematical activity; including conjecture and axiom formation, and evaluation and acceptance criteria. We hope to develop the automated case study in these aspects, as well as to extend concept formation to further mathematical domains.

6 Conclusion

We have argued that automatic concept formation is possible using analogous representations, and can lead to important and interesting mathematical concepts. We have demonstrated this via our automated case study. Finding a representation which enables either people or automated theory formation systems to fruitfully explore and develop a domain is an important challenge: one which we believe will be increasingly important, given the current focus in A.I. on embodied cognition.

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