Using analogies to find and evaluate mathematical conjectures

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Abstract. Analogical reasoning can shed light on both of the two key processes of creativity – generation and evaluation. Hence, it is a powerful tool for creativity. We illustrate this with three historical case studies of creative mathematical conjectures which were either found or evaluated via analogies. We conclude by describing our ongoing efforts to build computational realisations of these ideas.

1 Introduction

Analogical reasoning is an essential aspect of creativity [5; 19] and computational realisations such as [36; 27] have placed it firmly in the computational creativity arena. However, neither historical case studies of analogy nor computational representations tend to be in the domain of mathematics. We hold that this is an important domain since, as well as offering a great number of commonly accepted examples of creativity, it highlights features of analogical reasoning which may not be apparent in other areas. In this paper we show how mathematical analogies can aid *evaluation*, as well as the more obvious *generation*, of a mathematical conjecture, thus forming an essential tool in McGraw's "central loop of creativity" [24]. Given the fact that humans are not very good at making judgements, particularly in historically creative domains, any tool which aids evaluation is of particular interest. We also demonstrate the role of analogies in guiding re-representation. Recent work in analogy [2; 26] has suggested that current theories of analogical reasoning such as the structure mapping theory [13] may require some modification if they are to generalise to mathematics.

Focusing on analogy as a way of generating and evaluating ideas raises questions about how novel those ideas can be. By definition, an analogy-generated idea must share some sort of similarity with another domain, familiar to the creator. Thus the criterion of novelty, accepted as necessary for creative output (Boden characterised creativity as "the ability to generate novel and valuable ideas" [4]) seems to be under threat. In this paper we leave aside such considerations: since we consider the examples we give to be both unambiguously creative and unambiguously based on analogy, it seems that any definition of novelty must not exclude analogy (we hope to tackle this question in a future paper).

This paper comprises three historical case studies of analogy during creative phases in mathematics, and is thus largely theoretical, with the purpose of extending our understanding of analogies in creativity. In the penultimate section, however, we also describe some computational aspects of our work.

2 A marriage of dimensions

The analogy between different geometrical dimensions, and in particular between two and three dimensions dates back to Babylonian times [26] and has been particularly long lasting and productive. Polya called the analogy between plane and solid geometry "an inexhaustible source of new suggestions and new discoveries" [30, p. 26]. One example is the discovery of the Descartes–Euler conjecture, that for any polyhedron, the number of vertices (V) minus the number of edges (E) plus the number of faces (F) is equal to two. There are differing accounts of how this conjecture was discovered, but both involve analogy at some level. Euler's own account of his discovery suggests that analogy to two dimensional polygons was instrumental in guiding his search: "Recently it occurred to me to determine the general properties of solids bounded by plane faces, because there is no doubt that general theorems can be found for them, *just as for plane rectilinear figures* ..." (Euler in a letter to Christian Goldbach, Nov 1750 - our italics). The simple relationship that the number of edges of a two dimensional shapes is equal to the number of its vertices prompted a search for an analogous relationship between edges, faces and vertices in three dimensional solids (this can be written in the notation used for some analogies as $V = E : polygons :: ? : polyhedra$).

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In Polya's reconstruction of this discovery [30, pp. 35-41] he suggested that the analogy with two dimensions was introduced to *evaluate*, or *test* the conjecture, as opposed to *generate* it, as Euler implied. Polya argued that the conjecture was discovered via scientific induction, it passed increasingly difficult tests but failed on the example of the picture frame (a doughnut-shaped polyhedron for which $V - E + F = 16 - 32 + 16 = 0$). The conjecture was then modified to all *convex* polyhedra, and a new test was required with a new kind of support: analogy. Polya [30, pp. 42-43] developed a technique for using analogy to evaluate a conjecture: given analogical mappings and conjectures, Polya suggested that we adjust the representation in order to bring the relations closer. In the Descartes–Euler example, the re-representation works by noting that vertices are 0-dimensional, edges 1-dimensional, faces 2-dimensional and polyhedron 3-dimensional, and then rewriting both conjectures in order of the increasing dimensions. In the polygonal case, $V = E$ then becomes $V - E + 1 = 1$ (we could go one step further and rewrite the 1 as the number of faces, assuming that all polygons have one face), and the polyhedral case $V - E + F = 2$ becomes $V - E + F - 1 = 1$. These two equations now look much more similar: in both of them the number of dimensions starts at zero on the left hand side of the equation, increases by one and has alternating signs. The right hand side is the same in both cases. Polya then suggests that since the two relations are very close and the first relation, for polygons, is true, then we have reason to think that the second relation may be true, and is therefore worthy of a serious proof effort [29].

3 An extremely daring conjecture

The Basel problem is the problem of finding the sum of the reciprocals of the squares of the natural numbers, *i.e.* finding the exact value of the infinite series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots$$

In Euler's time this was a difficult and well known problem: it had been around at least since Pietro Mengoli posed it in 1650 [25], and had eluded efforts by Wallis, Leibniz and the Bernoulli brothers. Sandifer refers to it as "the best known problem of the time" [33, p. 58], and continues: "solving it would be as important as solving Fermat's Last Theorem was in 1993 or solving the Reimann Hypothesis would be today" *(ibid.)*. In this example then, the initial problem exists and analogy will be used to find a conjecture. Euler started working on the problem in 1730. In 1734 he found the exact value of $\frac{\pi^2}{6}$, solving the problem in three different ways in [8] and refining his solutions in [11] in 1740 and in [9; 10] in 1743. (For excellent commentary on this work, see [33, pp. 157-165] and [7, pp. 39-60].) We consider the third solution in [8], which is the usual one.

It was already known how to calculate the sum of finite series, and that a polynomial of degree $2n$, of the form:

$$a_0 - a_1x^2 + a_2x^4 - a_3x^6 + \ldots + (-1)^n b_n x^{2n} = 0, \tag{1}$$

with $2n$ different roots (values for which the equation holds) α_1, $-\alpha_1$, α_2, $-\alpha_2$, α_3, $-\alpha_3$,..., α_n, $-\alpha_n$ has the property that:

$$a_1 = a_0(\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \ldots + \frac{1}{\alpha_n^2}) \tag{2}$$

Euler considered the equation $sin\ x = 0$, which can also be written

$$\frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = 0$$

(where x ranges over $\mathbb{R}$). This has roots 0, π, $-\pi$, 2π, -2π, 3π, -3π, $\ldots$,. This is *quite* similar to the equation in *(1)*. In particular, if we disregard the root $x = 0$ and divide through by x, then we get:

$$\frac{sin\ x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots = 0. \tag{3}$$

This is now *very* similar to *(1)*, where $a_0 = 1, a_1 = \frac{1}{3!}, a_2 = \frac{1}{5!}, a_3 = \frac{1}{7!}$, etc., and $\alpha_1 = \pi$, $-\alpha_1 = -\pi$, $\alpha_2 = 2\pi$, $-\alpha_2 = -2\pi$, $\alpha_3 = 3\pi$, $-\alpha_3 = -3\pi$, ... The difference is that the equation in *(1)* is of finite degree, and has a finite number of roots, whereas the expansion in *(3)* is of infinite degree and has an infinite number of roots. If we disregard this difference, then we can rearrange *(3)* to look like the equation in *(2)*, *i.e.*,

$$\frac{1}{3!} = 1(\frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} \ldots).$$

This simplifies to $\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots$, and thus to Euler's solution of $\frac{\pi^2}{6}$ for the sum of the reciprocals of the squares.

Euler drew an analogy between finite and infinite series, and applied a rule about finite series to infinite series to get what is referred to by Polya as an "extremely daring conjecture" [30, p. 18]. Euler continued to evaluate this conjecture, and his analogous rule, by empirical tests: he calculated the value for more and more terms in the series, he used the rule on other infinite series to form predictions which could then be tested, both series with unknown solutions (for instance, $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\ldots = \frac{\pi^4}{90}$) and series where the solution was known (for instance, Leibniz's infinite series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots = \frac{\pi}{4}$). In all tests the conjecture and analogous rule held strong.

Let us summarise the role of analogy in this creative step. Euler saw the potential of using his knowledge about polynomials of degree n as a source domain. He knew the equivalence $sin\ x = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = 0$, and used it to get the roots $0, \pi, -\pi, 2\pi, -2\pi, 3\pi, -3\pi, \ldots$ He rearranged expressions in the target domain to make them more closely resemble expressions in the source domain, making sure that preconditions for the re-representation were satisfied (adding the constraint that $x \neq 0$). He knew that one aspect of the rule he had identified in the source domain failed to hold in the target (a finite number of terms and roots), and proceeded anyway with caution. This caution resulted in several different ways of evaluating both the conjecture and the application of the rule, and eventually to an independent proof of the conjecture. The original problem, to find an exact value for the sum of the reciprocals of the squares, was invented independently of the analogy which was used to solve it. Once the problem had been posed, Euler then formed the analogy to conjecture a solution. The conjecture was tested empirically and the analogy used to make predictions. If the rule gave the right answer for another infinite series, then Euler considered this to be evidence that it was correctly applied in the Basel case. Although finding a solution to the Basel problem was a major result in itself, the main value of Euler's work came from the discovery of the analogous rule, which - if confirmed - could be applied generally.

Generalising a rule from a source domain and applying it to a target domain, as Euler did, is one of the main characteristics of analogical reasoning. The rule may later be discovered to hold, fail, or be independent. While the modern mathematical concept of infinity was not developed in Euler's time, infinite series were an established concept (around at least since Zeno (*ca.* 490 - 430 B.C.), for instance $\sum_{n=1}^{\infty} \frac{1}{2^n}$ in Zeno's dichotomy [1]. Therefore, Euler's work with infinite series had to fit with the structure already developed. In the case we have described, the rule holds. In another example Polya [31, pp. 34, 221-222] describes a case in which the analogy between finite and infinite series fails. With finite series the order of the terms does not matter, hence terms in a finite series can be rearranged to an expression which may be easier to compute, but rearranging the order of terms in an infinite series may lead to a different result (in Polya's example he "proves" that for some sum l, $2l = l$). This is an example of a negative analogy (Hesse [18] used this term to describe analogies and mappings which suggested inferences subsequently found to be false). In some analogies, however, the target domain is much less developed and the analogiser may *define* the domain such that a rule which holds in a source domain shall also hold in the target. For example, in the development of arithmetic and the concept of number, addition on the rationals, negative numbers, irrationals, complex numbers and quaternions was defined as being analogous to addition on the natural numbers, such that the associative, commutative and distributive laws of algebra continued to hold. Likewise, the operations of addition/subtraction on the reals are analogous to multiplication/division: both are commutative and associative, both have an identity (though a different one) and both admit an inverse operation.

An analogiser may be unable to define a target domain such that a particular rule from the source holds. Consider for instance the development of quaternions. Hamilton wanted to develop a new type of number which was analogous to complex numbers but consisted of triples. While he could see a way of adding and subtracting these triples, he was unable to define multiplication[1]. He did discover a way of defining it for quadruples as

$$i^2 = j^2 = k^2 = ijk = -1.$$

However, his multiplication was non-commutative, although it was still associative and distributive. Alternatively, an analogiser may *wish* to define the target domain such that the rule from the source is broken. One example is the development by Martínez [23] of a system in which the traditional rule that $(-1)(-1) = +1$ is changed to $(-1)(-1) = -1$, resulting in a new mathematical system.

[1] One anecdotal source says that Hamilton wrote in a letter to his son: "Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edward and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."

4 A split in the real continuum

The history of the development of reasoning about real numbers throws up interesting issues for the development of mathematical theorising. Cauchy's conjecture and proof that "the limit of any convergent series of continuous functions is itself continuous" [6, p. 131] is another example in which a rule for one area is analogously assumed to hold for another area. In this case the source domain is *series* and the target *limits*, and the rule assumed to hold is that "what is true up to the limit is true at the limit". Lakatos [20, p. 128] states that throughout the eighteenth century this rule was assumed to hold and therefore any proof deemed unnecessary. However, this example is complicated: Cauchy's claim is generally regarded as obviously false, and the clarification of what was wrong is usually taken to be part of the more rigorous formalisation of the calculus developed by Weierstrass, involving the invention of the concept of *uniform convergence* (the historical route is sketched in [17, pp 213–17]).

This episode was treated by Lakatos in two different ways. Cauchy claimed that the function defined by pointwise limits of continuous functions must be continuous [6]. In fact, what we take to be counter-examples were already known when Cauchy made his claim, as Lakatos points out in his earlier analysis of the evolution of ideas involved [20, Appendix 1]. After discussion with Abraham Robinson, Lakatos then saw that there was an alternative analysis. Robinson was the founder of non-standard analysis, which found a way to rehabilitate talk of infinitesimals (for example, positive numbers greater than zero, but less than any "standard" real number (see [32], first edition 1966). Lakatos's alternative reading, presented in [21], is that Cauchy's proof was correct, but that his notion of (real) number was different from that adopted by mainstream analysis to this day. In analogy terminology, people who had different conceptions of the source domain were critiquing the target domain which Cauchy developed.

5 Computational considerations

We are currently exploring these ideas computationally in two ways. Firstly, we are using Lakoff and Núñez's notion of mathematical metaphor [22]. Lakoff and Núñez consider that the different notions of "continuum" outlined in §4 correspond to a discretised Number-Line blend (in the case of the Dedekind-Weierstrass reals); a discretised line as the result of "Spaces are Sets of Points" metaphor, where *all* the points on the line are represented (in the case where infinitesimals are present); or by a naturally continuous (physical) line [22, p. 288]. This approach provides promising avenues for the understanding of the relationships between the written representation of the mathematical theories, in this case mostly in natural language, the mathematical structures under consideration, and the geometrical or physical notions that informed the mathematical development. We are using the framework of Information Flow [3] to be more precise about what constitutes metaphors (and blends), by looking at the possible metaphorical relationships in terms of *infomorphisms* between domains. In [16; 15] we show how Information Flow theory [3] can be used to formalise the basic metaphors for arithmetic that ground the notions in embodied human experience (grounding metaphors). This gives us a form of implementation of aspects of the theory evolution involved here. We are extending this to Fauconnier and Turner's conceptual blending [12] and Goguen's Unified Concept theory [14].

Secondly, Schwering *et al.* have developed a mathematically sound framework for analogy making and a symbolic analogy model, heuristic-driven theory projection (HDTP) [34]. Analogies are established via a generalisation of source and target domain. Anti-unification [28] is used to compare formulae of source and target for structural commonalities, then terms with a common generalisation are associated in the analogical mapping. HDTP matches functions and predicates with same and different labels as well matching formulae with different structure. In particular, one of its features is a mechanism for re-representing a domain in order to build an analogy. We are currently involved in using this system to generate the domain of basic arithmetic from Lakoff and Núñez's four grounding metaphors.

6 Conclusion

We have described historical examples of analogy-generated and analogy-evaluated mathematical conjectures, and argued that mathematics is an important domain within which to develop notions of analogy in creativity. Analogical reasoning is a tool which aids both generation and evaluation in mathematics and is thus of great relevance to creativity researchers. Analogies are also used to find new concepts and entities (see [26] and [35, pp. 150-154]), as well as to *understand* conjectures; Lakatos showed this in [20], in which he advocated embedding a conjecture in a "possibly quite distant body of knowledge" [20, p. 9].

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