A Formal Cognitive Model of Mathematical Metaphors

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Abstract. Starting from the observation by Lakoff and Núñez (2000) that the process for mathematical discoveries is essentially one of creating metaphors, we show how Information Flow theory (Barwise & Seligman, 1997) can be used to formalise the basic metaphors for arithmetic that ground the basic concepts in the human embodied nature.

1 The Cognition of Mathematics

Mathematics is most commonly seen as the uncovering of eternal, absolute truths about the (mostly nonphysical) structure of the universe. Lakatos (1976) and Lakoff and Núñez (2000) argue strongly against the 'romantic' (Lakoff and Núñez) or 'deductivist' (Lakatos) style in which mathematics is usually presented.

Lakatos's philosophical account of the historical development of mathematical ideas demonstrates how mathematical concepts are not simply 'unveiled' but are developed in a process that involves, among other things, proposing formal definitions of mathematical concepts (his main example are polyhedra), developing proofs for properties of those concepts (e.g. Euler's conjecture that for polyhedra $V - E + F = 2$, where V, E, F are the number of vertices, edges and faces, respectively) and refining those concepts, e.g. by excluding counterexamples (what Lakatos calls *monster barring*) or widening the definition to include additional cases. Thus, the concept *polyhedron* changes in the process of defining its properties, and this casts doubt on whether ployhedra 'truly exist'.

Lakoff and Núñez (2000) describe how mathematical concepts are formed (or, depending on the epistemological view, how mathematical discoveries are made). They claim that the human ability for mathematics is brought about by two main factors: embodiment and the ability to create and use metaphors. They describe how by starting from interactions with the environment we build up (more and more abstract) mathematical concepts by processes of metaphor. They distinguish grounding from linking metaphors (p. 53). In *grounding* metaphors one domain is embodied and one abstract, e.g. the four grounding metaphors for arithmetic, which we describe below. In *linking* metaphors, both domains are abstract, which allows the creation of more abstract mathematical concepts, e.g. on the having established the basics of arithmetic with grounding metaphors this knowledge is used to create the concepts *points in space* and *functions* (p. 387).

* The research reported here was carried out in the *Wheelbarrow* project, funded by the EPSRC grant EP/F035594/1.

B. Mertsching, M. Hund, and Z. Aziz (Eds.): KI 2009, LNAI 5803, pp. 323–330, 2009.

Even if the universal truths view of mathematics were correct, the way in which people construct, evaluate and modify mathematical concepts has received little attention from cognitive science and AI. In particular, it has not yet been described by a computational cognitive model. As part of our research on understanding the cognition of creating mathematical concepts we are building such a model (Guhe, Pease, & Smaill, 2009; Guhe, Smaill, & Pease, 2009), based on the two streams of research outlined above. While there are cognitive models of learning mathematics (e.g. Lebiere 1998; Anderson 2007), there are to our knowledge no models of how humans *create* mathematics.

2 Structure Mapping, Formal Models and Local Processing

Following Gentner (1983; see also Gentner & Markman, 1997, p. 48) we assume that metaphors are similar to analogies. More specifically, there is no absolute distinction between metaphor and analogy; they rather occupy different but overlapping regions of the same space of possible mappings between concepts, with analogies comparatively mapping more abstract relations and metaphors more properties. According to Gentner's (1983, p. 156) structure mapping theory the main cognitive process of analogy formation is a mapping between the (higher-order) relations of conceptual structures.

Although we use this approach for creating computational cognitive models of mathematical discovery (see Guhe, Pease, & Smaill, 2009 for an ACT-R model using the path-mapping approach of Salvucci & Anderson, 2001), in this paper we use a formal model extending the one presented in Guhe, Smaill, and Pease (2009) that specifies the grounding metaphors for arithmetic proposed by Lakoff and Núñez (2000). Formal methods are not commonly used for cognitive modelling, but we consider it a fruitful approach to mathematical metaphors, because, firstly, mathematics is already formalised, and, secondly, if Lakoff and Núñez are correct that metaphors are a general cognitive mechanism that is applied in mathematical thinking, formal methods are an adequately high level of modelling this general purpose mechanism. Furthermore, collecting empirical data on how scientific concepts are created is difficult. This is true for case studies as well as laboratory settings. Case studies (e.g. Nersessian, 2008) are not reproducible (and therefore anecdotal), and they are usually created in retrospect, which means that they will contain many rationalisations instead of an accurate protocol of the thought processes. Laboratory settings in contrast (cf. Schunn & Anderson, 1998) require to limit the participants in their possible responses, and it is unclear whether or how this is different from the unrestricted scientific process.

Using the formal method, we take the metaphors used in the discovery process and create a formal cognitive model of the mapping of the high-level relations. The model is cognitive in that it captures the declarative knowledge used by humans, but it does not simulate the accompanying thought processes. For the formalisation we use Information Flow theory (Barwise & Seligman, 1997, see section 4), because it provides means to formalise interrelations (flows of information) between different substructures of a knowledge representation, which we called *local contexts* in Guhe (2007). A local context contains a subset of the knowledge available to the model that is relevant with respect to a particular *focussed element*. In cognitive terms, a focussed element is an item (concept, percept) that is currently in the focus of attention.

The ability to create and use suitable local contexts is what sets natural intelligent systems apart from artificial intelligent systems and what allows them to interact efficiently and reliably with their environment. The key benefit of this ability is that only processing a local context (a subset of the available knowledge) drastically reduces the computational complexity and enables the system to interact with the environment under real-time constraints.

3 Lakoff and Núñez's Four Basic Metaphors of Arithmetic

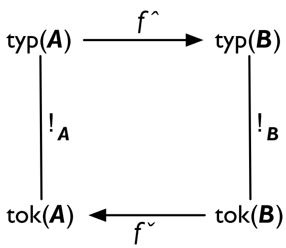
Lakoff and Núñez (2000, chapter 3) propose that humans create arithmetic with four different grounding metaphors that create an abstract conceptual space from embodied experiences, i.e. interactions with the real world. Since many details are required for describing these metaphors adequately, we can only provide the general idea here. Note that the metaphors are not interchangeable. All are used to create the basic concepts of arithmetic.

1. **Object Collection.** The first metaphor, *arithmetic is object collection*, describes how by interacting with objects we experience that objects can be grouped and that there are certain regularities when creating collections of objects, e.g. by removing objects from collections, by combining collections, etc. By creating metaphors (analogies), these regularities are mapped into the domain of arithmetic, for example, collections of the same size are mapped to the concept of number and putting two collections together is mapped to the arithmetic operation of addition.
2. **Object Construction.** Similarly, in the *arithmetic is object construction* metaphor we experience that we can combine objects to form new objects, for example by using toy building blocks to build towers. Again, the number of objects that are used for the object construction are mapped to number and constructing an object is mapped to addition.
3. **Measuring Stick.** The *measuring stick* metaphor captures the regularities of using measuring sticks for the purpose of establishing the size of physical objects, e.g. for constructing buildings. Here numbers correspond to the physical segments on the measuring stick and addition to putting together segments to form longer segments.
4. **Motion Along A Path.** The *motion along a path* metaphor, finally, adds concepts to arithmetic that we experience by moving along straight paths. For example, numbers are point locations on paths and addition is moving from point to point.

4 Information Flow

This section gives a short introduction to Information Flow theory. We focus on the aspects needed for our formalisation, see Barwise and Seligman (1997) for a detailed discussion. We need three notions for our purposes: *classification*, *infomorphism* and *channel*.

Classification. A *classification* $\boldsymbol{A}$ consists of a set of tokens $\text{tok}(\boldsymbol{A})$, a set of types $\text{typ}(\boldsymbol{A})$ and a binary classification relation $\models_{\boldsymbol{A}}$ between tokens and types. In this way, the classification relation classifies the tokens, for example, for a token $a \in \text{tok}(\boldsymbol{A})$ and a type $\alpha \in \text{typ}(\boldsymbol{A})$ the relation can establish $a \models_{\boldsymbol{A}} \alpha$. Graphically, a classification is usually depicted as in the left of figure 1, i.e. with the types on top and the tokens on the bottom.



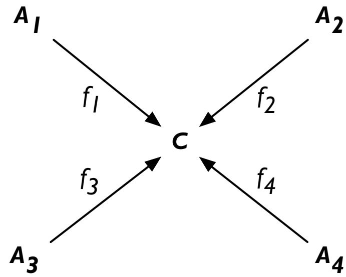


Fig. 1. Left: Two classifications (***A*** and ***B***) and an infomorphism (f) in Information Flow; Right: Channel $C = \{f_1, f_2, f_3, f_4\}$ and its core $\boldsymbol{C}$

Infomorphism. An *infomorphism* $f : \boldsymbol{A} \rightleftarrows \boldsymbol{B}$ from a classification $\boldsymbol{A}$ to a classification $\boldsymbol{B}$ is a (contravariant) pair of functions $f = \langle f\hat{}, f\check{} \rangle$ that satisfies the following condition:

$$f\check{}(b) \vDash_{\boldsymbol{A}} \alpha \quad \text{iff} \quad b \vDash_{\boldsymbol{B}} f\hat{}(\alpha)$$

for each token $b \in \text{tok}(\boldsymbol{B})$ and each type $\alpha \in \text{typ}(\boldsymbol{A})$, cf. figure 1.

Note that the type relation $f\hat{}$ and the token relation $f\check{}$ point in opposite directions. A mnemonic is that the ˆ of $f\hat{}$ points upwards, where the types are usually depicted.

Channel. A *channel* is a set of infomorphisms that have a common codomain. For example, the channel C shown in figure 1 consists of a family of four infomorphisms f_1 to f_4 that connect the four classifications $\boldsymbol{A}_1$ to $\boldsymbol{A}_4$ to the common codomain $\boldsymbol{C}$. The common codomain is the *core* of the channel. Note that the infomorphisms are all pairs of functions, i.e. $f_1 = \langle f_1\hat{}, f_1\check{} \rangle$, etc.

The core is the classification that contains the information connecting the tokens in the classifications $\boldsymbol{A}_1$ to $\boldsymbol{A}_4$. For this reason, the tokens of $\boldsymbol{C}$ are called *connections*. In our application to arithmetic the core is the arithmetic knowledge that represents what is common to the different source domains.

Channels and cores are the main way in which Information Flow achieves a distributed, localised representation of knowledge. In other words, this is the property of the Information Flow approach that fits to the localised representation and processing found in cognition. At the same time, infomorphisms provide a principled way of representing the connections between the different local contexts.

5 Formalisation of the Metaphors for Arithmetic

The basic idea of how to apply Information Flow theory to the four basic metaphors of Lakoff and Núñez (2000) is that each domain (object collection, object construction, measuring stick, motion along a path and arithmetic) is represented as a classification and the metaphors between the domains are infomorphisms.

Information Flow captures regularities in the distributed system (Barwise & Seligman, 1997, p. 8). So, the infomorphisms between the four source domains and the core (arithmetic) capture the regularities that link these domains to arithmetic and the knowledge shared by these domains. (A full arithmetic classification contains more than these commonalities – think of arithmetic concepts arising by linking metaphors like the concept of zero – but for our current purposes it suffices to think of it this way.)

In Guhe, Smaill, and Pease (2009) we present the object collection and object construction metaphors. We will, therefore, present the remaining two metaphors here. Note that we are only formalising the basics of the metaphors here. However, we chose the formalisation with respect to the required extensions, e.g. the iteration extension of the measuring stick metaphor, which should be rather straightforward.

5.1 Measuring Stick

The measuring stick metaphor, cf. table 1, is formalised with vectors, which facilitate the extensions proposed by Lakoff and Núñez (2000), e.g. for the metaphors creating the notion of square roots (p. 71). We define a classification ***MS*** where the tokens of the measuring stick domain are actual physical instances of measuring sticks encountered by the cognitive agent. We name the elements $\vec{s}_A$, $\vec{s}_B$, etc. The tokens are classified by their length, i.e. types are a set equivalent to the set of natural numbers $\mathbb{N}$. The classification relation $\vDash_{\boldsymbol{MS}}$ classifies the measuring sticks by their length. Given this classification, we can now assign a type to each token, e.g. $\vec{s}_A \vDash_{\boldsymbol{MS}} 3, \vec{s}_B \vDash_{\boldsymbol{MS}} 5$.

Table 1. The measuring stick metaphor

measuring stick	arithmetic
physical segments	numbers
basic physical segment	one
length of the physical segment	size of the number
longer	greater
shorter	less
acts of physical segment placement	arithmetic operations
putting physical segments together end to end	addition
taking a smaller physical segment from a larger one	subtraction

- **Physical segments (consisting of parts of unit length):** A physical segment (of parts of unit length) is a vector (of multiples of the unit vector, i.e. the vector of length 1). All vectors in this classification are multiples of the unit vector and all vectors have the same orientation and direction.
- **Basic physical segment:** The basic physical segment is a vector $\vec{a}$ with $|\vec{a}| = 1$.
- **Length of physical segment:** The length of a physical segment is the length of the vector: $length_{\boldsymbol{MS}}(\vec{s}) = |\vec{s}|$.
- **Longer/shorter:**

$$longer_{\boldsymbol{MS}}(\vec{s}_A, \vec{s}_B) = \begin{cases} true, & \text{if } length_{\boldsymbol{MS}}(\vec{s}_A) > length_{\boldsymbol{MS}}(\vec{s}_B), \\ false, & \text{if } length_{\boldsymbol{MS}}(\vec{s}_A) \leq length_{\boldsymbol{MS}}(\vec{s}_B) \end{cases}$$

  $shorter_{\boldsymbol{MS}}$ is defined analogously.

- **Acts of physical segment placement:** Operations on vectors. Results of the segment placements (operations) are physical segments (vectors).
- **Putting physical segments together end-to-end with other physical segments:** This operation is defined as vector addition:

$$putTogether_{\boldsymbol{MS}}(\vec{s}_A, \vec{s}_B) = \vec{s}_A + \vec{s}_B$$

As all vectors have the same orientation, the length of the vector created by the vector addition is the sum of the lengths of the two original vectors.
- **Taking shorter physical segments from larger physical segments to form other physical segments:** This operation is also defined as vector addition, but this time the shorter vector ($\vec{s}_B$) is subtracted from the longer vector ($\vec{s}_A$):

$$takeSmaller_{\boldsymbol{MS}}(\vec{s}_A, \vec{s}_B) = \vec{s}_A - \vec{s}_B$$

As above, the vectors have the same orientation, so the lengths 'add up'. The corresponding type is the difference of the two lengths.

5.2 Motion along a Path

The motion along a path metaphor, cf. table 2, is formalised as sequences (lists), which we write as $\langle \dots \rangle$. The elements of the sequences are $\bullet$ (any symbol will do here). The classification is called ***MP***. The sequences are named $path_A$, $path_B$, etc. Again the tokens are classified by their length; the type set is $\mathbb{N}$.

- **Acts of moving along the path:** Operations on sequences.
- **The origin, the beginning of the path:** The empty sequence: $\langle\rangle$. Because $\mathbb{N}$, with which model the arithmetic domain, does not contain 0, origins are not part of the model. We consider this the first (very straightforward) extension of the basic metaphors, which is reflected in the fact that historically 0 was a rather late invention.
- **Point-locations on a path:** Sequences of particular lengths, and the result of operations on sequences. The length of a sequence is given by the function $length_{\boldsymbol{MP}}$, which returns the number of elements in the sequence.
- **The unit location, a point-location distinct from the origin:** The sequence with one element: $\langle\bullet\rangle$.

Table 2. The arithmetic is motion along a path metaphor

motion along a path	arithmetic
acts of moving along the path	arithmetic operations
a point location on the path	result of an operation; number
origin; beginning of the path	zero
unit location, a point location distinct from the origin	one
further from the origin than	greater
closer to the origin than	less
moving away from the origin a distance	addition
moving toward the origin a distance	subtraction

- **Further from the origin than/closer to the origin than:**

$$further_{\boldsymbol{MP}}(path_A, path_B) = \begin{cases} true, \text{ if } length_{\boldsymbol{MP}}(path_A) > length_{\boldsymbol{MP}}(path_B), \\ false, \text{ if } length_{\boldsymbol{MP}}(path_A) \leq length_{\boldsymbol{MP}}(path_B) \end{cases}$$

  $closer_{\boldsymbol{MP}}$ is defined analogously.
- **Moving from a point-location *A* away from the origin, a distance that is the same as the distance from the origin to a point-location *B*:** Moving a distance *B* away from a point *A* that is different from the origin is the concatenation of the sequences:

$$moveAway_{\boldsymbol{MP}}(path_A, path_B) = path_A \circ path_B$$

  where $\circ$ concatenates two sequences in the usual fashion:

$$\langle \bullet_1 \bullet_2, \ldots \rangle \circ \langle \bullet_I, \bullet_{II}, \ldots \rangle = \langle \bullet_1 \bullet_2, \ldots, \bullet_I, \bullet_{II}, \ldots \rangle$$

  The corresponding type is the addition of the two lengths.
- **Moving toward the origin from *A*, a distance that is the same as the distance from the origin to *B*:** For this operation to be possible, it must be the case that $length(path_A) > length(path_B)$.

$$towardOrigin_{\boldsymbol{MP}}(path_A, path_B) = path_C, \text{where } path_A = path_B \circ path_C$$

  The corresponding type is the difference of the two lengths.

5.3 Arithmetic

The classification ***AR*** representing arithmetic consists of the abstract natural numbers as tokens and concrete instances of natural numbers as types. The smallest number is 1. The arithmetic operations are defined as usual[1].

5.4 Metaphor Infomorphisms and Channel

Given these three classifications we can now define the channel *C* as the set of two infomorphisms ($C = \{f_{\boldsymbol{MS}}, f_{\boldsymbol{MP}}\}$) between the two classifications representing the source domains (***MS*** and ***MP***) and the core ***AR***.

Infomorphism from Measuring Stick to Arithmetic. The infomorphism linking the measuring stick collection domain to arithmetic is defined as $f_{\boldsymbol{MS}} : \boldsymbol{MS} \rightleftarrows \boldsymbol{AR}$. The relation between types is then $f\hat{}_{\boldsymbol{MS}}(n) = length_{\boldsymbol{MS}}(\vec{s}_A)$, where $n \in \mathbb{N}$ and $length_{\boldsymbol{MS}}(\vec{s}_A) = n$, the relation between tokens $f\check{}_{\boldsymbol{MS}}(n) = \vec{s}_A$ where $n \in \mathbb{N}$ and $\vec{s}_A$ is a representation of the physical measuring stick that the number refers to.

The shortest measuring stick is $f\check{}_{\boldsymbol{MS}}(\vec{s}_A) = 1$, where $length_{\boldsymbol{MS}}(\vec{s}_A) = 1$ and the comparisons and operations are mapped as $f\hat{}_{\boldsymbol{MS}}(longer_{\boldsymbol{MS}}) = >$, $f\hat{}_{\boldsymbol{MS}}(shorter_{\boldsymbol{MS}}) = <$, $f\hat{}_{\boldsymbol{MS}}(+) = +$, $f\hat{}_{\boldsymbol{MS}}(-) = -$. Recall that the types for $putTogether_{\boldsymbol{MS}}$ and $takeSmaller_{\boldsymbol{MS}}$ are the sum/difference of the lengths of the vectors representing the tokens.

[1] Note that types can be structured. The natural way to look at >, for example, is to look at the sum of two classifications, each of which is some version of number: $\mathbb{N} \rightarrow \mathbb{N} + \mathbb{N} \leftarrow \mathbb{N}$. The tokens of $\mathbb{N} + \mathbb{N}$ are pairs of tokens (tk_1, tk_2) from the two copies, and the type that judges whether one token is 'greater' than the other returns a Boolean dependent on the token components tk_1, tk_2.

Infomorphism from Motion along a Path to Arithmetic. Similar to the definition above, the informorphism $f_{MP} : MP \rightleftarrows AR$ relates types as $f\hat{}_{MP}(n) = length_{MP}(path_A)$, where $n \in \mathbb{N}$ and $length_{MP}(path_A) = n$, and tokens as $f\check{}_{MP}(n) = path_A$, where $n \in \mathbb{N}$ and $path_A$ represents a path being referred to by the number. The other properties are defined as: $f\hat{}_{MP}(bigger_{MP}) = >$, $f\hat{}_{MP}(smaller_{MP}) = <$, $f\hat{}_{MP}(+) = +$, $f\hat{}_{MP}(-) = -$. As above, for $putTogether_{MP}$ and for $takeSmaller_{MP}$ the corresponding types are the sum/difference of the lengths of the sequences representing the tokens.

6 Conclusions and Future Work

Based on the work by Lakatos (1976) and by Lakoff and Núñez (2000) we argued that mathematics is not the uncovering of unchanging, eternal truths, but at its core it is a cognitive process. This means, that even if such truths exist, humans still discover them using cognitive processes. We demonstrated how Information Flow can be used to formalise the basic metaphors for arithmetic, i.e. to create a high-level cognitive model of these metaphors, and presented two of them (see Guhe, Smaill, & Pease, 2009 for the other two). We are currently working on an implementation of this formalisation that additionally uses quotient classifications (Barwise & Seligman, 1997, sec. 5.2).

References

Anderson, J.R.: How can the human mind occur in the physical universe? Oxford University Press, Oxford (2007)

Barwise, J., Seligman, J.: Information flow: The logic of distributed systems. Cambridge University Press, Cambridge (1997)

Gentner, D.: Structure-mapping: A theoretical framework for analogy. Cognitive Science 7(2), 155–170 (1983)

Gentner, D., Markman, A.B.: Structure mapping in analogy and similarity. American Psychologist 52(1), 45–56 (1997)

Guhe, M.: Incremental conceptualization for language production. Erlbaum, Mahwah (2007)

Guhe, M., Pease, A., Smaill, A.: A cognitive model of discovering commutativity. In: Proc. of the 31st Annual Conf. of the Cognitive Science Society (2009)

Guhe, M., Smaill, A., Pease, A.: Using information flow for modelling mathematical metaphors. In: Proc. of 9th Intern. Conf. on Cognitive Modeling (2009)

Lakatos, I.: Proofs and refutations: The logic of mathematical discovery. Cambridge University Press, Cambridge (1976)

Lakoff, G., Núñez, R.E.: Where mathematics comes from: How the embodied mind brings mathematics into being. Basic Books, New York (2000)

Lebiere, C.: The dynamics of cognition: An ACT-R model of cognitive arithmetic. Unpublished doctoral dissertation, CMU Computer Science Department (1998)

Nersessian, N.J.: Creating scientific concepts. MIT Press, Cambridge (2008)

Salvucci, D.D., Anderson, J.R.: Integrating analogical mapping and general problem solving: The path-mapping theory. Cognitive Science 25(1), 67–110 (2001)

Schunn, C.D., Anderson, J.R.: Scientific discovery. In: Anderson, J.R., Lebiere, C. (eds.) The atomic components of thought, pp. 385–427. Erlbaum, Mahwah (1998)